MINIMAL SUPERSYMMETRIC EXTENSION OF THE STANDARD MODEL OF ELEMENTARY PARTICLES A RENORMALIZATION GROUP APPROACH

By

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Abstract of Dissertation Presented to the Graduate School
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MINIMAL SUPERSYMMETRIC EXTENSION OF THE
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A RENORMALIZATION GROUP APPROACH

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This work describes how the renormalization group technique can be utilized to probe the physics that may lie beyond the standard model, of elementary particles. In particular, we shall focus on the so-called minimal extension of the standard model which is a very good candidate for what could become the "new standard model." After reviewing the tenets of the renormalization group in the context of the standard model, we apply its probing abilities to the minimal supersymmetric extension of the standard model. We thus obtain very quickly some rather restrictive but consistent bounds on the masses of the as yet undiscovered top quark and Higgs boson predicted by the standard model. We then dwell further on the intricate Yukawa sector of the theory to see if the renormalization group may shed some light on its more intimate structure in the form of relations satisfied by the parameters of the theory at the scale where the electromagnetic, weak, and strong interactions in nature seem to unify. We find that supersymmetry is indeed a viable idea for extending the standard model in a phenomenological manner compatible with all the known experimental facts and the theoretically appealing idea of unification and that the renormalization group is a tool of choice for such an endeavor.

CHAPTER I
INTRODUCTION

At this point in time the standard model of elementary particles seems to account remarkably for the precision physics performed in the accelerator facilities worldwide. In fact, we could say that it works too well! After all, telltale signs of the physics that may lie beyond the standard model have yet to be observed. Proton decay, rare kaon decays, or even neutrino masses (despite some uncertain experimental discrepancies [1]) are such examples. We would be happy, albeit admittedly rather bored as physicists, if the standard model were the end of the story (at least up to Planck scale 10^{19} GeV where gravity would come into the picture). However, we have some reason, mostly theoretical, to believe that some new physics should manifest itself somewhere between the electroweak scale (100 GeV) and Planck scale (10^{19} GeV) and the good news is that it could be much closer to the former than the latter, typically around a TeV. On theoretical ground the standard model is not entirely satisfactory; there are too many free parameters, too many elementary particles in multiple generations, no understanding of the pattern of quarks and leptons, a huge (gauge) hierarchy of scales [2], "unnatural" fine-tuning to keep the mass of the Higgs scalar of the order of the electroweak scale rather than the Planck scale [3], and, of course, no gravity. The source of belief in a scale of a TeV for some new physics has to do with trying to understand electroweak symmetry breaking. In the standard model the Higgs boson is a scalar point-like particle that is responsible for the electroweak symmetry breaking

that gives masses to the massive gauge bosons the quarks and the leptons. But the "naturalness" problem does not make it an entirely satisfying picture. Possible extensions that could help involve modifying our understanding of the electroweak symmetry breaking. This can be done dynamically, in which case new strong interaction dynamics in analogy with the strong force in QCD break the electroweak symmetry, as in technicolor [3,4] where the Higgs is now a condensate of so-called technoquarks. Or it can be accomplished by introducing new weakly coupled dynamics associated with the electroweak symmetry breaking, as in supersymmetry [5,6,7] where the approximate symmetry between boson and fermion keep the Higgs "natural." Both extensions need to introduce the cure at a scale of a TeV, a scale that is beginning to be accessible experimentally.

The approach for divining the possible extensions of the standard model is twofold. In the first, the quantum numbers of the standard model are grouped into mathematically pleasing structures resulting in an exercise in quantum number pattern recognition. This has led to the early grand unified theories (GUT) [8,9,10,11]. The second approach [12] is to use the renormalization group to extrapolate the standard model parameters to the unexplored scales. The purpose is to find if those parameters satisfy interesting relations at shorter distances. When used in conjunction with the former approach, this can give powerful hints of the physics expected at shorter scales. Of course it all depends on having accurate data to input as initial conditions on the renormalization group equations as well as a strong theoretical basis for the evolution equations themselves. The minimal SU(5) GUT is the prototype for such analyses [13]. There, properties of the model at energies of order 10^{15} GeV are translated with the aid of the renormalization group to a prediction

for proton decay that is not consistent with experiment. Data on the coupling constants are now sufficiently precise to rule out most simple GUTs, including minimal SU(5) because of the absence of unification of the running couplings (GUT triangle) [14,15,16]. The same analysis has recently improved the feasibility of the supersymmetric extensions of these GUTs. Renormalization group methods are of enduring practical importance in the attempts of high energy physicists to glean indications of more fundamental theories from radiative corrections.

We therefore collect in a comprehensive manner and in one place the necessary tools for making renormalisation group analyses of the standard model and of its various extensions [17]. With the ever increasing precision of experiment, we expect the inclusion of two-loop effects to be crucial. Therefore we work to two-loop order and, moreover, include the complete Yukawa sector contribution, as well as treat properly the effects of thresholds. This constitutes an improvement over previous efforts. We use numerical methods to evolve the parameters to different scales using the β functions found in the literature [18] and plot the results for representative values of the Higgs boson and top quark masses.

Most of the parameters in the standard model are to be found in the Yukawa sector of the theory where they parametrize quark and lepton masses, the interfamily mixings of the quarks and CP violation. Historically, only one of these thirteen parameters was ever predicted [19], the charmed quark mass but only after an inspired guess on the value of a strong (i.e., presently incalculable) matrix element. This vanishing of the "GUT triangle" of the gauge couplings, hinting at a supersymmetric GUT, should be matched by concomitant simplicity in the other parameters of the theory. To that purpose we

present a comparative analysis of possible relations among Yukawa couplings at shorter distances [20] both in the standard model itself and in its minimal supersymmetric extension [21,22].

While we find little evidence to support the view that the standard model is by itself the low energy manifestation of a pure GUT, we are encouraged by the results of this investigation. Inclusion of supersymmetry allows many possible GUT relations among Yukawa couplings to be satisfied at one appropriate gauge unification scale.

Promoting supersymmetry to a local symmetry results in supergravity which as its name suggests is an attempt to include gravity in an all encompassing gauge theory. It can be viewed as a low energy limit of a so-called "theory of everything" namely superstring theory. Understanding of this theory requires looking in particular at two dimensional field theories because of the assumed stringy nature of particles. There has been therefore some interest in the so-called matrix model understanding of certain matter system coupled to two dimensional quantum gravity. These models can be described in terms of a zero-curvature Lax-pair formulation with therefore a connection to the integrable hierarchies that appear in integrable systems with infinitely many degrees of freedom.

We have therefore looked into the construction of Lax operators to describe the standard hierarchies. We studied the approach of Drinfel'd and Sokolov [23] based on affine Lie algebras. We found a diagrammatic construction of the Lax operators that describes the well known modified KdV hierarchies based on all the affine Lie algebras [24]. In fact we found that we can associate to every irreducible representation of any affine Lie algebra a Lax operator and therefore an integrable hierarchy. The method of construction is typically

faster than the procedure of Drinfel d and Sokolov and is based on the weight diagram construction of the unique highest weight vector associated to every irreducible representation of an affine Lie algebra The weight diagram displays the weight vectors obtained by the action of the lowering operators of the algebra on the highest weight vector Finally, since our method was general enough we extended our scheme to deal with supersymmetric hierarchies and found that some modifications were required to accommodate the subtleties of supersymmetric affine Lie algebras but a diagrammatic approach can still be undertaken. This method should allow us to describe supersymmetric hierarchies not yet discussed in the literature

The outline of this work is as follows. In the next chapter we summarize how we use the renormalization group throughout this work, doing so in the context of the standard model [17]

In chapter 3 we apply these techniques to the minimal supersymmetric extension of the standard model and develop some immediate consequences for the masses of the top quark and Higgs boson [21]

Still using the renormalization group, we continue exploring the Yukawa sector of the standard model in chapter 4 and for its minimal supersymmetric extension in chapter 5 [20] There we consider patterns inscribed in relations between the parameters of the theory, relations motivated by theory or otherwise, which could be valid at higher energy scale to glean some hindsight on the physics that may lie beyond the standard model

In chapters 6 through 9 we discuss the diagrammatic construction of Lax operators [24] In chapter 6 we review the necessary background material on Lie algebras and Lax operators The construction proper of these Lax operators is considered in chapter 7 both in the original way and in our new

diagrammatic approach. We then prove the validity of our construction in chapter 8. Finally, in chapter 9 we show how this method is generalized in the supersymmetric case.

In appendices I to III we have included some technical aspects of the study of the renormalization group for the standard model and its minimal supersymmetric extension, mostly about the β functions used in this work [17]. In the last appendix we describe some work done by the author on a self dual solution in $2+1$ gravity and its connection with its Chern Simmons description [25].

CHAPTER 2
THE STANDARD MODEL

2.1 Renormalization Procedures

Renormalization is a reparametrization of a theory which renders Green functions and physical quantities finite order by order, in perturbation theory. A specific choice of renormalized parameters defines a renormalization scheme. The physics is, of course, independent of how the theory is renormalized. A common way of relating bare and renormalized parameters is

$$g_0 = g - \delta g \ , \tag{2.1.1}$$

where g_0 is the bare parameter, g is the renormalized parameter, and δg is the counterterm. Fixing the counterterms by requiring them to consist only of the infinite terms needed to render the theory finite, defines the minimal subtraction (MS) prescription [26]. A feature of the MS scheme is a mass scale μ which enters in the process of regularizing divergent integrals using dimensional regularization. Furthermore the unit of mass μ is used to keep couplings dimensionless when continuing to d dimensions in the dimensional regularization procedure. For example, if Eq. (2.1.1) represents any of the three gauge couplings of the standard model, then μ is introduced as follows to keep them dimensionless

$$g_0(g\mu)^{-\epsilon} = g - \delta g \ , \tag{2.1.2}$$

where ϱ is a constant parametrizing the arbitrariness in the finite parts of divergent integrals in dimensional regularization and $\epsilon = (4-d)/2$. Equation (2.1.2) defines a family of MS schemes. Choosing $\varrho = 1$ is the simplest MS scheme that was described above. Choosing $\varrho^2 = e^{\gamma_E}/4\pi$, where $\gamma_E = 0.5722\ldots$ is the Euler-Mascheroni constant, defines the so-called modified minimal subtraction ($\overline{\rm MS}$) prescription [27]. This scheme is the most commonly employed in QCD calculations, and it is the one we adopt. The free parameters of the standard model in the $\overline{\rm MS}$ schemes are μ dependent. Their μ evolution is governed by the β functions of the renormalization group. Moreover, these running parameters are not in general equal to their corresponding physical values (consequently for the masses, we adopt a convention wherein upper case Ms refer to physical values and lower case ms denote $\overline{\rm MS}$ values). This is to be contrasted with the on-shell renormalization scheme in which, for example, the renormalized masses equal their physical values and the renormalized electromagnetic coupling equals the fine structure constant. However, the MS schemes have the attractive characteristic that the β functions are μ independent and therefore particularly simple to integrate. Physical quantities $P(\{g_i(\mu)\},\mu)$ expressed in terms of μ and the running parameters of the theory, $\{g_i(\mu)\}$, must be μ independent

$$\mu\frac{d}{d\mu}P(\{g_i(\mu)\},\mu) = (\mu\frac{\partial}{\partial\mu} + \beta_i\frac{\partial}{\partial g_i})P = 0 \qquad (2.1.3)$$

where the β_i are the β functions. The two-loop β functions of the standard model have been collected in Appendix I.

As mentioned above physical quantities are renormalization scheme independent. However, this assumes that calculations can be done without approximation. In reality, calculations are only perturbative approximations and these do depend on the renormalization scheme. In QCD where the strong

coupling α_s is large, there will be renormalization scheme dependence problems. In the electroweak model as in QED where the couplings are small this is not so great a problem.

2.2 Initial Value Extraction from Data

$\alpha_1(M_Z)$ and $\alpha_2(M_Z)$

The determination of the $SU(2)_L \times U(1)_Y$ couplings proceeds from the standard model relations:

$$\begin{aligned} \alpha_1(\mu) &\equiv \frac{g_1'^2(\mu)}{4\pi} = C^2 \frac{\alpha(\mu)}{\cos^2\theta_W(\mu)} , \\ \alpha_2(\mu) &\equiv \frac{g_2^2(\mu)}{4\pi} = \frac{\alpha(\mu)}{\sin^2\theta_W(\mu)} \end{aligned} \tag{2.2.1}$$

where $\alpha(\mu) = e^2(\mu)/4\pi$ and C^2 is a normalization constant that equals 1 for the standard model and equals $\frac{5}{3}$ when the standard model is incorporated in grand unified theories of the SU(N) and SO(N) type [12]. What is required to specify these couplings are the values of $\alpha(\mu)$ and $\sin^2\theta_W(\mu)$ in the renormalization scheme we employ (α_{em} $\overline{\rm MS}$). The electromagnetic fine structure constant ($\alpha_{em}^{-1} \approx 137.036$) is extrapolated from zero momentum scale to a scale μ equal to M_Z in our case. The renormalized coupling $\alpha(\mu)$ is related to the fine structure constant α_{em} as follows:

$$\alpha_{em} = \frac{\alpha(\mu)}{1 + \Pi(0)} , \tag{2.2.2}$$

where $\Pi(q^2)$ is the renormalized vacuum polarization function. In the standard model where there are many species of charged fermions and charged gauge bosons, Eq. (2.2.2) generalizes to [28]

$$\alpha^{-1}(\mu) = \alpha_{em}^{-1} - \frac{2}{3\pi} \sum_f Q_f^2 (\ln \frac{\mu}{m_f}) \theta(\mu - m_f) + \frac{1}{6\pi} . \tag{2.2.3}$$

The effects of the strong interaction that enter as a hadronic contribution to the vacuum polarization function must be included also This yields a value

$$\alpha^{-1}(M_Z) = 127.9 \pm 0.3 \tag{2.2.4}$$

The process independent, renormalized weak mixing angle $\sin^2\hat{\theta}_W$ of the on shell scheme is defined to be

$$\sin^2\hat{\theta}_W \equiv 1 - \frac{M_W^2}{M_Z^2}\ , \tag{2.2.5}$$

where M_W and M_Z are the physical masses of the W and Z gauge bosons. Knowing the precise values of the W and Z boson masses and using the equation above provides one way of extracting the value of $\sin^2\hat{\theta}_W$ Alternatively, the bare relation involving the low energy Fermi constant measured in muon decay and the W boson mass

$$\frac{G_{\mu_e}}{\sqrt{2}} = \frac{e_0^2}{8\sin^2\hat{\theta}_{W_0} M_{W_0}^2} \tag{2.2.6}$$

may be corrected to order α

A third way of extracting $\sin^2\hat{\theta}_W$ is from neutral current experiments, among which deep inelastic neutrino scattering appears to provide the best determination A running $\sin^2\theta_W(\mu)$ may be defined in $\overline{\rm MS}$ and differs from the above $\sin^2\hat{\theta}_W$ by order α corrections The $\overline{\rm MS}$ running W boson mass $m_W(\mu)$ and the corresponding physical mass M_W, identified as the simple pole at $q^2 = M_W^2$ of the W propagator are related as follows

$$M_W^2 = m_W^2(\mu) + A_{WW}^T(M_W^2, \mu)\ , \tag{2.2.7}$$

where $A_{W\,W}^T$ is the transverse part of the W self energy A similar relation holds for the Z boson In $\overline{\rm MS}$ renormalization the following relation defines the running $\sin^2\hat{\theta}_W(\mu)$

$$\sin^2\hat{\theta}_W(\mu) = 1 - \frac{m_W^2(\mu)}{m_Z^2(\mu)} \tag{2.2.8}$$

Equation (2.2 7) and its Z analog may be combined with Eq (2 2 6) to extract $\sin^2\theta_W$ A fit to all neutral current data gives

$$\sin^2\theta_W(M_Z) = 0\ 2324 \pm 0\ 0011\ , \tag{2.2 9}$$

for arbitrary M_t [29] Using these values of $\alpha(M_Z)$ and $\sin^2\theta_W(M_Z)$ yields

$$\begin{aligned} \alpha_1(M_Z) &= 0.01698 \pm 0\ 00003 \\ \alpha_2(M_Z) &= 0\ 03364 \pm 0\ 0002 \end{aligned} \tag{2.2 10}$$

$\alpha_s(M_Z)$

The value of the strong coupling is known with less precision than most of the parameters of the standard model. This is due to large theoretical uncertainties arising from the nonperturbative nature of low energy QCD and the slow convergence of perturbation series in high energy QCD Moreover, this uncertainty is hard to quantify

In the extraction of $\alpha_s = g_s^2/4\pi$ from a physical process many obstacles arise Since the convergence of the QCD perturbation theory series is not very fast, one must check higher order effects Even if one chooses processes that do not involve hadronization, a most delicate problem in the extraction of α_s comes from working to finite order in perturbation theory Physical quantities should of course be renormalization scheme independent, but the necessity of approximation introduces dependence on the renormalization scheme Typically, the same physical quantity calculated in two different schemes to the n^{th} order of α_s will differ by terms of order α_s^{n+1} As α_s is large, this difference may be large and thus may lead to renormalization scheme dependence problems This problem manifests itself in the difficulty of choosing the renormalization scale μ to use for the particular experiment from which one is extracting the strong coupling Ideally one would like to choose μ to minimize the unknown

higher order terms but that is of course not possible Sometimes μ is approximated by the scale at which the highest order known term vanishes or the scale at which that term gives a stationary prediction However, the most frequent choice is $\mu = E$, where E is some characteristic energy scale of the experiment This choice is plausible since it minimizes the typical terms that arise that involve $\ln(Q/\mu)$, with Q some momentum in the process, typically $\sim E$ All the processes from which the strong coupling is extracted suffer from this problem and thus each individual extraction of α_s has large uncertainties To obtain the best estimate of the strong coupling we shall take together the results from different processes These include e^+e^- scattering into hadrons heavy quarkonium decay scaling violations in deep inelastic lepton hadron scattering and jet production in e^+e^- scattering

To summarize [17] the values of $\alpha_s(M_Z)$ and its error are given in Table I

Table I. Values of α_s at M_Z and its error

Process	α_s	$\Delta\alpha_s$
$e^+e^- \to$ hadrons	0 140	0 018
Υ decay	0 111	0 005
Deep inelastic scattering	0 109	0 008
Jet distribution in e^+e^-scattering	0 115	0 008

To pick the value of $\alpha_s(M_Z)$ for our numerical studies we take the Gaussian weighted average of these values $([\sum(\alpha_s/\Delta\alpha_s^2)/\sum(1/\Delta\alpha_s^2)] \pm [\sum(1/\Delta\alpha_s^2)]^{-\frac{1}{2}})$ and we find [30] $\alpha_s(M_Z) = 0\ 113 \pm 0\ 004$

Yukawa Couplings

To take full account of the Yukawa sector in running all the couplings initial values for the Yukawa couplings are necessary They must be extracted from physical data such as quark masses and CKM mixing angles Furthermore, the interesting parameters to be plotted must be determined step by step in the process of running to Planck mass These two procedures are not unrelated and require the diagonalization of the up type down type, and leptonic Yukawa matrices

The interaction Lagrangian for the Yukawa sector is (following Machacek and Vaughn s convention [18])

$$\mathcal{L} = \overline{Q}_L \Phi \mathbf{Y}_u^\dagger u_R + \overline{Q}_L \tilde{\Phi} \mathbf{Y}_d^\dagger d_R + \overline{\ell}_L \Phi \mathbf{Y}_e^\dagger e_R + h.c., \tag{2.2.11}$$

where flavor indices have been suppressed and where Q_L and ℓ_L are the quark and lepton SU(2) doublets, respectively

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{2.2.12}$$

The fields Φ and $\tilde{\Phi}$ are the Higgs scalar doublet and its SU(2) conjugate

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \tilde{\Phi} = \tau_2 \Phi^* \tag{2.2.13}$$

u_R, d_R and e_R are the quark and lepton SU(2) singlets, and $\mathbf{Y}_{u,d,e}$ are the matrices of the up-type down type and lepton type Yukawa couplings

After spontaneous symmetry breaking the following mass terms are obtained

$$\mathcal{L}_{mass} = \frac{v}{\sqrt{2}} (\overline{u}_L \mathbf{Y}_u^\dagger u_R + \overline{d}_L \mathbf{Y}_d^\dagger d_R + \overline{e}_L \mathbf{Y}_e^\dagger e_R) + h.c \tag{2.2.14}$$

The fields appearing in $\mathcal{L}_{mass}$ are actually the gauge transformed fields of $\mathcal{L}$ The diagonalization is implemented through the following bi unitary transfor mations

$$u_L \to S_u u_L, \quad u_R \to T_u u_R,$$
$$d_L \to S_d d_L, \quad d_R \to T_d d_R, \tag{2.2 15}$$
$$e_L \to S_e e_L \quad e_R \to T_e e_R$$

with

$$u = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \; d = \begin{pmatrix} d \\ s \\ b \end{pmatrix} \; e = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \tag{2.2 16}$$

and $S_{u\,d\,e}$ and $T_{u,d\,e}$ are unitary 3×3 matrices. Under these transformations the mass terms are diagonalized

$$\mathcal{L}_{mass} \to \frac{v}{\sqrt{2}}[\overline{u}_L(S_u{}^{\dagger}\mathbf{Y}_u{}^{\dagger}T_u)u_R + \overline{d}_L(S_d{}^{\dagger}\mathbf{Y}_d{}^{\dagger}T_d)d_R + \overline{e}_L(S_e{}^{\dagger}\mathbf{Y}_e{}^{\dagger}T_e)e_R] + h.c. \tag{2.2 17}$$

Let Δ_u denote the diagonal Yukawa matrix for the up type quarks,

$$\Delta_u = \frac{\sqrt{2}}{v}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \tag{2.2.18}$$

Therefore

$$S_u{}^{\dagger}\mathbf{Y}_u{}^{\dagger}T_u = \Delta_u \tag{2.2 19}$$

and

$$T_u{}^{\dagger}\mathbf{Y}_u S_u = \Delta_u^{\dagger} = \Delta_u, \tag{2.2 20}$$

so that

$$S_u{}^{\dagger}(\mathbf{Y}_u{}^{\dagger}\mathbf{Y}_u)S_u = \Delta_u^2 \tag{2.2.21}$$

Similarly

$$S_d{}^{\dagger}(\mathbf{Y}_d{}^{\dagger}\mathbf{Y}_d)S_d = \Delta_d^2, \quad S_e{}^{\dagger}(\mathbf{Y}_e{}^{\dagger}\mathbf{Y}_e)S_e = \Delta_e^2 \tag{2.2.22}$$

These transformations leave all kinetic terms and weak neutral current interaction terms unaffected; however in the hadronic charged current interactions they are responsible for quark flavor mixing:

$$J_\mu^+ \sim \overline{Q}_L \gamma_\mu \tau^+ Q_L = \overline{u}_L \gamma_\mu d_L \to \overline{u}_L \gamma_\mu V d_L \tag{2.2.23}$$

where $V = S_u^\dagger S_d$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. It is a unitary 3×3 matrix that is often parametrized as follows:

$$V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix} \tag{2.2.24}$$

where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$, $i = 1, 2, 3$.

The entries of the parametrized CKM matrix can be related simply to the experimentally known CKM entries. The particle data book [29] gives the following ranges of values (assuming unitarity) for the magnitudes of the elements of the CKM matrix:

$$|V| = \begin{pmatrix} 0.9747 - 0.9759 & 0.218 - 0.224 & 0.001 - 0.007 \\ 0.218 - 0.224 & 0.9734 - 0.9752 & 0.030 - 0.058 \\ 0.003 - 0.019 & 0.029 - 0.058 & 0.9983 - 0.9996 \end{pmatrix} \tag{2.2.25}$$

These ranges of values can be converted to bounds for s_i, $i = 1, 2, 3$ and $\sin\delta$. We arrive at these bounds by finding values for the four angles such that the entries of the CKM matrix obtained from these satisfy the conditions imposed by Eq. (2.2.25). We find:

$$\begin{aligned} 0.2188 \leq \sin\theta_1 \leq 0.2235 \\ 0.0210 \leq \sin\theta_2 \leq 0.0543 \\ 0.0045 \leq \sin\theta_3 \leq 0.0290 \end{aligned} \tag{2.2.26}$$

However, the accuracy with which $|V|$ is known does not constrain $\sin\delta$. A set of angles $\{\theta_1, \theta_2, \theta_3, \delta\}$ was chosen that falls within the ranges quoted above. The initial data needed to run the Yukawa elements are extracted from

the CKM matrix and the quark masses A problem arises though for the mixing angles, which was solved for the quark masses (see next subsection), in that it is not clear at what scale the chosen initial values for these angles should be considered known However, we have observed that for the whole range of initial values, the running of the mixing angles is quite flat, with a perceptible increase in θ_2 between M_W and the Planck scale for higher top masses This is in accordance with the angles being related to ratios of quark $m$asses and therefore the exact knowledge of that scale (or lack thereof) is not as critical as might be feared *a priori*

The diagonalization procedure outlined above must be reversed to extract the initial Yukawas Therefore Eq (2 2 19) and its analogue for d and e yield

$$\begin{aligned} \mathbf{Y}_e &= T_e \Delta_e S_e^\dagger \\ \mathbf{Y}_d &= T_d \Delta_d S_d^\dagger, \\ \mathbf{Y}_u &= T_u \Delta_u S_u^\dagger \end{aligned} \tag{2 2 27}$$

There is freedom in the choice of the S and T matrices in generating the $\mathbf{Y}$ matrices Since in the standard model there are no right handed charged currents, the right handed flavor rotation matrices $T_{u,d,e}$ can be set to the identity freely From the form of the CKM, its unitarity can be maintained simply by setting either S_u or S_d equal to the identity We chose the latter Hence the initial Yukawas can be written in terms of the physical parameters in the following manner

$$\begin{aligned} \mathbf{Y}_e &= \Delta_e = \frac{\sqrt{2}}{v} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \\ \mathbf{Y}_d &= \Delta_d = \frac{\sqrt{2}}{v} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \\ \mathbf{Y}_u &= \Delta_u V = \frac{\sqrt{2}}{v} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} V \end{aligned} \tag{2 2 28}$$

Known Quark Masses

As QCD is assumed to imply quark confinement, extraction of quark masses from experiment follows the same circuitous route as other QCD quantities such as α_s. In the past decade a variety of techniques have been developed and utilized to extract quark masses from the observed particle spectrum.

The light u and d quark masses are the ones least accurately known. They are determined by a combination of chiral perturbation techniques and QCD spectral sum rules (QSSR). In the former case the light quark masses are directly expressible in terms of the parameters of the explicit SU(2) and SU(3) chiral symmetry breakings. Together with the ratio [31]

$$\frac{m_s}{m^1} = 25.7 \pm 2.6 \tag{2.2.29}$$

they imply the following renormalization group invariant mass ratio

$$\frac{m_d - m_u}{2m^1} = 0.28 \pm 0.03 \tag{2.2.30}$$

Spectral sum rules in QCD are obtained in an attempt to relate the observed low energy spectrum to the parameters describing the high energy domain where perturbation theory becomes applicable to the quark gluon picture [32,33,34,35]. Applied to the light quarks these sum rules imply [31]

$$m_u + m_d = 24.0 \pm 2.5 \text{ MeV} \tag{2.2.31}$$

Together with Eq. (2.2.30) they reduce to

$$\begin{aligned} m_u &= 8.7 \pm 0.8 \text{ MeV}, \\ m_d &= 15.4 \pm 0.8 \text{ MeV} \end{aligned} \tag{2.2.32}$$

The parameter $\hat{m}$ is a renormalization group invariant, which to three loops, is related to the $\overline{\rm MS}$ running mass parameter $m(\mu)$ via [36]

$$m(\mu) = m\Big[-\beta_1\frac{\alpha_s(\mu)}{\pi}\Big]^{-\gamma_1/\beta_1}\Big\{1+\frac{\beta_2}{\beta_1}\Big(\frac{\gamma_1}{\beta_1}-\frac{\gamma_2}{\beta_2}\Big)\frac{\alpha_s(\mu)}{\pi} + \frac{1}{2}\Big[\frac{\beta_2^2}{\beta_1^2}\Big(\frac{\gamma_1}{\beta_1}-\frac{\gamma_2}{\beta_2}\Big)^2-\frac{\beta_2^2}{\beta_1^2}\Big(\frac{\gamma_1}{\beta_1}-\frac{\gamma_2}{\beta_2}\Big)+\frac{\beta_3}{\beta_1}\Big(\frac{\gamma_1}{\beta_1}-\frac{\gamma_3}{\beta_3}\Big)\Big]\Big[\frac{\alpha_s(\mu)}{\pi}\Big]^2\Big\}\,, \tag{2.2.33}$$

where the β_i and the γ_i are the coefficients of the β functions for α_s and m given in Appendix I. From Eqs. (2.2.33) (to two loops) and (2.2.32) one may infer the following values

$$\begin{aligned} m_u(1\ \text{GeV}) &= 5.2 \pm 0.5\ \text{MeV}\,, \\ m_d(1\ \text{GeV}) &= 9.2 \pm 0.5\ \text{MeV} \end{aligned} \tag{2.2.34}$$

The strange quark mass is determined, averaging the value derived from Eqs. (2.2.29) and (2.2.31) with those obtained using Eq. (2.2.32) and the various QSSR values for $m_u + m_s$ to be [37]

$$m_s = 266 \pm 29\ \text{MeV}\,, \tag{2.2.35}$$

corresponding to the running value

$$m_s(1\ \text{GeV}) = 154 \pm 4\ \text{MeV} \tag{2.2.36}$$

For the heavier quarks: charm and bottom, one can make a more precise prediction. Here, the nonrelativistic bound state approximation may be applied. The physical mass $M(q^2 = M^2)$ can be extracted from the running mass at the pole mass to three loops via

$$m(q^2 = M^2) = \frac{M(q^2 = M^2)}{1 + \frac{4}{3}\frac{\alpha_s(M)}{\pi} + K(\frac{\alpha_s(M)}{\pi})^2}\,, \tag{2.2.37}$$

where $K = 13.3$ for the charm and $K = 12.4$ for the bottom quarks [38]. We run (to three loops) the masses and α_s to obtain the following values at the conventionally preferred scale of 1 GeV [17]

$$m_c(1\text{ GeV}) = 1.41 \pm 0.06\text{ GeV}$$
$$m_b(1\text{ GeV}) = 6.33 \pm 0.06\text{ GeV} \tag{2.2.38}$$

Our numerical approach does not make any more approximations than the ones assumed in the β functions and the mass equations used, apart from the approximation inherent in the numerical method itself, and, we therefore believe, is more in line with our program than using the "perturbatively integrated" form of the β functions. Thus we shall adopt the above values. It should be stressed that at the low scales under consideration the three-loop α_s corrections we have included in our mass and strong coupling β functions are often comparable to the two-loop ones and hence affect the accuracy of our final values noticeably.

Lepton Masses

The physical (pole) masses of the leptons are very well known [29]

$$M_e = 0.51099900 \pm 0.00000015\text{ MeV}$$
$$M_\mu = 105.658357 \pm 0.000034\text{ MeV} \tag{2.2.39}$$
$$M_\tau = 1.7841^{+0.0027}_{-0.0030}\text{ GeV}$$

We use these values to determine initial data for the running masses. Some authors neglect QED corrections and use the physical values for the running values at $\sim M_Z$, which introduces only a small error. By calculating the one-loop self-energy corrections, one arrives at a QED relation between the running $\overline{\text{MS}}$ masses and the corresponding physical masses

$$m_l(\mu) = M_l[1 - \frac{3\alpha(\mu)}{4\pi}(1 + 6\frac{\mu^2}{m_l^2} + \frac{4}{3})] \tag{2.2.40}$$

Choosing $\mu = 1$ GeV as in the quark mass case and using Eqs. (2.2.40) and (2.2.3) yields the running lepton masses (taking $m_l = M_l$ in the log term above is an appropriate approximation to order α)

$$\begin{aligned} m_e(1\text{ GeV}) &= 0.4800\text{ MeV}\,, \\ m_\mu(1\text{ GeV}) &= 104.57\text{ MeV} \\ m_\tau(1\text{ GeV}) &= 1.7835\text{ GeV} \end{aligned} \tag{2.2.41}$$

Higgs Boson and Top Quark Masses

The Higgs boson and top quark masses have not been measured directly at present. Consistency with experimental data on $\sin^2\theta_W$ requires $M_t < 197$ GeV for $M_H = 1$ TeV at 99% CL assuming no physics beyond the standard model [16]. Precision measurements of the Z mass and its decay properties combined with low energy neutral current data have been used to set stringent bounds on the top quark mass within the minimal standard model. A global analysis of this data yields $M_t = 122^{+41}_{-32}$ GeV for all allowed values of M_H [39]. Recent experimental lower bound $M_t \gtrsim 91$ GeV. As for the Higgs, the analysis of Ref. 39 gives the restrictive bound, $M_H \lesssim 600$ GeV, if $M_t < 120$ GeV and $M_H < 6$ TeV, for all allowed M_t. Since perturbation theory breaks down for $M_H \gtrsim 1$ TeV, the latter bound on the Higgs boson mass is not necessarily meaningful. LEP data set a lower bound on the Higgs boson mass of 48 GeV [40].

In our analysis, initial values of the $\overline{\rm MS}$ running top quark mass m_t and of the scalar quartic coupling λ at M_Z are chosen arbitrarily (consistent with the bounds quoted above). As noted earlier, these running parameters are not equal to their physical counterparts. However, any reasonable prediction for the masses of the top quark and of the Higgs boson that may come from our

analysis should be that of experimentally relevant physical masses. Therefore formulas similar to Eq. (2.2.40) relating $\overline{\rm MS}$ running parameters to physical masses are needed. To calculate the physical or pole mass of the top quark we use Eq. (2.2.37) in its general form

$$\frac{M_t}{m_t(M_t)} = 1 + \frac{4}{3}\frac{\alpha_s(M_t)}{\pi} + [16.11 - 1.04\sum_{i=1}^{5}(1 - \frac{M_i}{M_t})](\frac{\alpha_s(M_t)}{\pi})^2 \ , \quad (2.2.42)$$

where M_i, $i = 1, \ldots, 5$, represent the masses of the five lighter quarks. Likewise the physical mass of the Higgs boson can be extracted from the following relation [41]

$$\lambda(\mu) = \frac{G_\mu}{\sqrt{2}} M_H^2 (1 + \delta(\mu)) \ , \quad (2.2.43)$$

where $\delta(\mu)$ contains the radiative corrections. Its form is rather elaborate and we relegate it to Appendix III. Equations (2.2.42) and (2.2.43) are highly non-linear functions of M_t and M_H, respectively. Their solution requires numerical routines.

Vacuum Expectation Value of the Scalar Field

The vacuum expectation value (vev) of the scalar field may be extracted from the well known lowest order relation

$$v = (\sqrt{2} G_\mu)^{-\frac{1}{2}} = 246.22 \text{ GeV} \quad (2.2.44)$$

From the very well measured value of the muon lifetime, $\tau_\mu = 2.197035 \pm .000040 \times 10^{-6}$ s [29], the Fermi constant can be extracted using the following formula [42]

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} f(\frac{m_e^2}{m_\mu^2})(1 + \frac{3}{5}\frac{m_\mu^2}{m_W^2})[1 + \frac{\alpha(m_\mu)}{2\pi}(\frac{25}{4} - \pi^2)] \ , \quad (2.2.45)$$

where

$$f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \ . \quad (2.2.46)$$

giving

$$G_\mu = 1.16637 \pm 0.00002 \times 10^{-5}\ \mathrm{GeV}^{-2} \tag{2.2.47}$$

This parameter may be viewed as the coefficient of the effective four-fermion operator for muon decay in an effective low energy theory

$$\frac{G_\mu}{\sqrt{2}}[\overline{\nu}_e\gamma^\beta(1-\gamma_5)e][\overline{\mu}\gamma_\beta(1-\gamma_5)\nu_\mu] \tag{2.2.48}$$

A direct calculation (e.g., in the Landau gauge) of the electromagnetic corrections yields that the operator is finitely renormalized (i.e. G_μ does not run) [43,44]. Another way to see this is by using a Fierz transformation to rewrite the above expression

$$\frac{G_\mu}{\sqrt{2}}[\overline{\nu}_e\gamma^\beta(1-\gamma_5)\nu_\mu][\overline{\mu}\gamma_\beta(1-\gamma_5)e] \tag{2.2.49}$$

The neutrino current does not couple to the photon field, and the $e-\mu$ current is conserved and is hence not multiplicatively renormalized.

We need an initial value for the running vacuum expectation value at some scale μ. Wheater and Llewellyn Smith [45] consider muon decay to order α in the context of the full electroweak theory and derive an equation relating an $\overline{\mathrm{MS}}$ running G_μ to the experimentally measured value. From this formula we can extract a value for $v(M_Z)$. However, the formula is derived in the 't Hooft-Feynman gauge, and the evolution equation Eq. (1.18) of Appendix I for the vev is valid only in the Landau gauge. Nevertheless, motivated by the discussion of the previous paragraph, we choose the initial condition for the vev to be $v(M_H) = 246.22$ GeV. Using an appropriate initialization algorithm, we arrive at $v(M_Z)$. We find that this procedure leads to no significant correction, and we therefore take, *ab initio*, $v(M_Z) = 246.22$ GeV.

2.3 Threshold Analysis

We are using the $\overline{\rm MS}$ renormalization scheme to determine the running of the standard model couplings. From the Appelquist Carrazzone decoupling theorem [46] we expect the physics at energies below a given mass scale to be independent of the particles with masses higher than this threshold. However such minimal subtraction schemes are not physical in the sense that they are scale dependent and mass independent so that the decoupling theorem is not manifest. As described in Refs. 47 and 48 we have to formulate a low energy effective theory by integrating out the heavy fields to one loop and find matching functions in order to take care of the threshold effects.

The starting point for treating thresholds in MS and $\overline{\rm MS}$ schemes is the construction of low energy effective gauge theories [47,49]. The basic idea is to integrate out the heavy fields in such a way that the remaining effective action is gauge invariant under the residual gauge group. Let the simple gauge group G be broken to $\tilde{G}$, and let Φ and ϕ be a set of heavy and light fields, respectively. Then the action $\tilde{I}[\phi]$ of the effective field theory is obtained from the action $I[\phi\ \Phi]$ of the full theory by functional integration over the heavy fields

$$e^{i\tilde{I}[\phi]} = \int [d\Phi] e^{iI[\phi\ \Phi]} \tag{2.3.1}$$

Because there are no superheavy fields in the effective theory the decoupling theorem is not needed. However there is a difficulty having to do with gauge invariance. Namely in order to integrate out the heavy fields one has to add a gauge fixing term, and such a term usually spoils the gauge invariance of the low energy theory. The usual R_ξ gauge fixing action for the full gauge group is

$$I_{gf} = -\frac{1}{2}\sum_a \int d^4x\, f_a^2(w,\phi,\Phi)\ . \tag{2.3.2}$$

where, for example,

$$f_\alpha = \xi^{-1/2}[\partial_\mu A^\mu_\alpha + ig\xi v t_{\alpha S} S] \,, \qquad (2.3\,3)$$

where α runs over all generators $t_{\alpha S}$ of G. A^μ_α are the gauge fields, and S is a column of Hermitian scalar fields with vacuum expectation value v. However, when the heavy gauge fields are integrated out with the natural choice of the gauge fixing functional as in (2.3.3) but summed only over the index of the broken group generators (given in uppercase Latin letters), one obtains an effective theory that is not invariant under $\tilde{G}$. We correct this by changing the derivative in (2.3.3) into a $\tilde{G}$ covariant derivative so that f becomes

$$f_A = \xi^{-1/2}[\partial_\mu A^\mu_A + gC_{AB\alpha} A^\mu_B A_{s\mu} + ig\xi v t_{AS} S] \,, \qquad (2.3.4)$$

where the lowercase Latin indices run over the generators of the unbroken gauge group and $C_{\alpha\beta\gamma}$ are the completely antisymmetrized structure constants of G.

In application to the standard model [17] we will integrate out the heavy fields using matching functions. As we are running couplings to two loops, it is sufficient to integrate out these fields to one loop. The heavy gauge bosons, their ghosts and the top quark are integrated out near M_W, the other fermions at their physical masses M_f. One may integrate out different mass particles at one scale as long as the two-loop contribution to coupling constant renormalization between two threshold scales is negligible. The errors arising from not integrating out fields at a scale μ exactly equal to their physical mass M is of order $a/12\ln(M/\mu)$ which is negligible within the perturbative regime.

2.4 Quantitative Analysis

We depict the results of numerically integrating the β functions for the standard model parameters from 1 GeV to Planck mass in Figs. 1 to 3. For

most of these plots, we have made the arbitrary choice $M_t = M_H = 100$ GeV, to illustrate the main features In Fig 1 we display the evolution of the inverse of each of the three gauge couplings, including the associated uncertainties in their values We see the "GUT triangle" signifying the absence of grand unification, assuming the standard model as an effective theory in the desert up to the Planck scale We note, as is well known, that the uncertainties do not fill in the "GUT triangle."

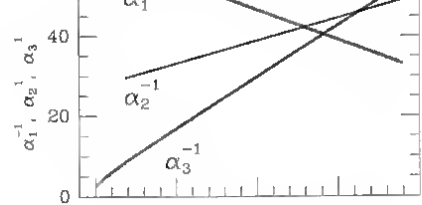


Figure 1- Running of the inverse gauge couplings using their propagated experimental errors for the two-loop case only

In Fig 2, we plot the quartic coupling λ and the top Yukawa coupling y_t for ($M_t = 100$ GeV $M_H = 100$ GeV) and for ($M_t = 200$ GeV, $M_H = 195$ GeV)

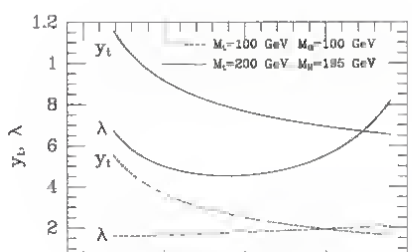


Figure 2 Top Yukawa and scalar quartic couplings.

These two couplings are the only unknown parameters of the standard model. We have studied the effects of changing the values of M_t and M_H in our analyses of the running of the other parameters. We observed that for any M_t between 100 GeV and 200 GeV, varying M_H while maintaining perturbativity and vacuum stability, did not affect appreciably the evolution of any of the other parameters. However, changing M_t itself showed a significant difference in the running of the heavier quarks. In particular we observed that the intersection point between the bottom quark and the τ lepton moves down to a lower scale for a higher top quark mass. This is expected since from Eq. (I.9) one can see that the bottom type Yukawas are driven down by an increased top Yukawa. This is to be contrasted with the SUSY GUT case, as we shall

see in the next chapter, where the bottom Yukawa β function is such that this crossing point is shifted toward a higher scale with an increased top mass. In an SU(5) SUSY GUT model, the equality of the bottom and τ Yukawas at the scale of unification can be used to get bounds on the top and Higgs masses. Lastly, we display the running of the CKM angles in Fig. 3. We are using the initial data: $\sin\theta_1 = 0.2206$, $\sin\theta_2 = 0.0298$, and $\sin\theta_3 = 0.0106$. Also we have taken $\delta = 90°$, which corresponds to the case of maximal CP violation. As mentioned in Section 2.4, the evolution curves for these angles are effectively flat.

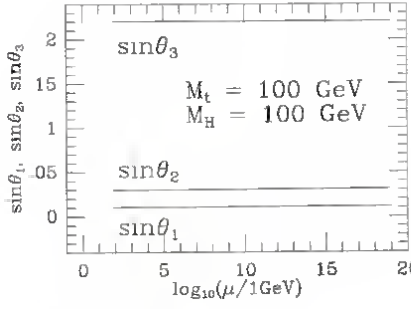


Figure 3 Running of the CKM angles

The largest differences between one-loop and two-loop evolution occur in the bottom, charm, and strange quark masses. In the present case of the standard model we find that two-loop running of the parameters does at times improve on the one loop running. Indeed, we have tabulated the differences of several parameters in their one versus two-loop values at various scales, for the cases ($M_t = 100$ GeV $M_H = 100$ GeV) and ($M_t = 200$ GeV, $M_H = 195$ GeV). Table II illustrates the difference one-loop and two-loop running makes in the ratio m_b/m_τ, for the three scales 10^2 GeV, 10^4 GeV, and 10^{16} GeV.

Table II m_b/m_τ

	$M_t = 100$ GeV			$M_t = 200$ GeV		
	10^2 GeV	10^4 GeV	10^{16} GeV	10^2 GeV	10^4 GeV	10^{16} GeV
1 loop	1.879	1.455	0.8081	1.868	1.392	0.6647
2 loop	1.782	1.348	0.7336	1.769	1.255	0.6047

Clearly the difference between one- and two-loop results is more pronounced at higher scales, as expected. Over all these scales the difference is never less than 5.5%. We note that the ratio becomes equal to one well below the scale of grand unification. The next table presents a similar comparison for the top Yukawa. Here, two loops represent a smaller correction with the difference at all scales always being less than 5%.

Table III y_t						
	$M_t = 100$ GeV			$M_t = 200$ GeV		
	10^2 GeV	10^4 GeV	10^{16} GeV	10^2 GeV	10^4 GeV	10^{16} GeV
1-loop	0 5405	0 4160	0 1928	1 133	0 9780	0 7145
2-loop	0 5405	0 4071	0 1842	1 143	0 9700	0 6816

Finally, Table IV displays the same analysis for α_s for the case $M_t = M_H = 100$ GeV. We observe no appreciable deviation from the tabulated values for any $M_t \lesssim 200$ GeV (except in the low energy regime where the difference is at most $\sim 4\%$)

Table IV α_s				
	1 GeV	10^2 GeV	10^4 GeV	10^{16} GeV
1 loop	0 3128	0 1118	0 07103	0 02229
2/3 loop	0 3788	0 1117	0 07030	0 02208

At scales $\lesssim M_Z$ the inclusion of two loops is important in the evolution of the strong coupling (and of the quark masses). Indeed, we find that the pure QCD three-loop contribution is also significant and therefore include it in the running of the strong coupling and of the quark masses in the low energy region. As seen in this table, the combined two and three loops in the low energy regime account for a 17% difference at 1 GeV in α_s.

Although in the cases considered in these last two tables there does not appear to be a significant difference in two-loop over one-loop evolution at

scales above M_Z, the first table does show a 10% difference at the scale 10^{16} GeV. We expect two-loop effects to be more important when the theory is extended, e.g. to include supersymmetry and/or grand unification.

The effects of using a naive step approximation rather than a proper treatment of thresholds are numerically unimportant for the cases discussed above. Indeed they are less important than the two-loop effects. We note however that the inclusion of nonnaive thresholds effects is significant in the numerical analysis of extensions of the standard model.

CHAPTER 3
THE MINIMAL SUPERSYMMETRIC EXTENSION OF THE SM

We now present bounds on the mass of the top quark in a minimal supersymmetric extension of the standard model (MSSM) with minimal Higgs structure in the context of a grand unified theory (GUT) by numerically evolving the couplings using their renormalization group equations. This analysis improves on previous endeavors by taking full account of the Yukawa sector.

3.1 Motivation

The $\overline{\text{MS}}$ renormalization group equations for the standard model and the MSSM [18] are numerically integrated and used to evolve the parameters of the model to Planck scale. Although it is not possible to analytically express certain parameters (e.g. CKM angles) in terms of the Yukawa couplings, equations for the running of the quantities themselves can be arrived at by making some approximations. Typically one assumes that the contribution of the Yukawa couplings is essentially given by the top quark one- y_t since it is larger than the others. Otherwise one can retain the effects of the other Yukawas by keeping only the diagonal entries, but since our approach is numerical we opt to run the quantities by diagonalizing the Yukawa matrices at every step of the Runge-Kutta method.

In the expectation that the standard model is only the low energy manifestation of some yet unknown GUT or of a possible supersymmetric (SUSY)

extension thereof, the three couplings g_3, g_2, and g_1 corresponding to the standard model gauge groups, $SU(3)^C \times SU(2)^W \times U(1)^Y$, should meet at some large grand unification scale. Our study begins here. Using the accepted values and associated errors of these couplings we observe unification in the SUSY GUT case but not in the pure GUT case, as noted by several groups [14,15,16] (see Fig. 4)

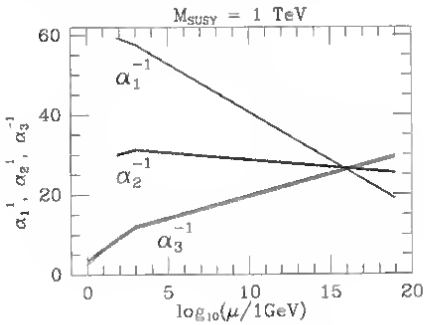


Figure 4 Plot of the running of the inverse couplings. The dotted lines above and below the solid lines represent the experimental error for each coupling. The window depicts a blow up of the area around the unification point. (Note the small region where all three couplings intersect. We found that this region reduced to a point when $M_{SUSY} = 8.9$ TeV and was non existent above that scale.

However, this should not be viewed as proof of supersymmetry since given the values of α_1, α_2, α_3 at some scale and three unknowns (the value of α at the unification scale, the unification scale, and an extra scale such as the SUSY scale) there is always a solution. The exciting aspect of the analysis of Ref. 14 is the numerical output, namely a low SUSY scale, M_{SUSY}, and a perturbative solution below the Planck scale which does not violate proton decay bounds.

Furthermore, in the context of a minimal GUT [9] there are constraints on the Yukawa couplings at the scale of unification. In this work we first restrict ourselves to an SU(5) SUSY GUT [50] where y_b and y_τ, the bottom and τ Yukawa couplings, are equal at unification. The crossing of these renormalization group flow lines is sensitive to the physical top quark mass, M_t. This can be seen in the down type Yukawa renormalization group equation (above M_{SUSY} for example) from which we extract the evolution of y_b, since the top contribution is large and appears already at one loop through the up-type Yukawa dependence

$$\begin{aligned}\frac{d\mathbf{Y}_d}{dt} =& \frac{1}{16\pi^2}\mathbf{Y}_d[\,3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_u{}^\dagger\mathbf{Y}_u + \mathrm{Tr}\{3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} \\ & -(\frac{7}{15}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2)\,]\end{aligned} \tag{3.1.1}$$

where $\mathbf{Y}_{u\,d,e}$ are matrices of Yukawa couplings. Demanding that their crossing point be within the unification region determined by the gauge couplings allows one to constrain M_t. This yields an upper and lower bound for M_t which nevertheless is fairly restrictive.

3.2 Method

Let us now briefly describe our method [17]. We work in a mass independent renormalization scheme where the running couplings are unphysical.

From the decoupling theorem [46] we expect the physics at energies below a given mass scale to be independent of the particles with masses higher than this threshold. Therefore for a correct interpretation of these running couplings we must take into account the thresholds [51,48]. For the electroweak threshold we use one loop matching functions [48] with the two loop beta functions valid in the standard model regime below the SUSY scale. These matching functions are obtained in $\overline{\rm MS}$ renormalization by integrating out the heavy gauge fields in such a way that the remaining effective action is invariant under the residual gauge group [47]. At the electroweak threshold, near M_W, we integrate out the heavy gauge fields and the top quark. Below this threshold there is an effective $SU(3)^c \times U(1)^{EM}$ theory. Thresholds in this region are obtained by integrating out each quark to one loop at a scale equal to its physical mass. At these scales the one loop matching functions to the gauge couplings vanish and the threshold dependence appears through steps in the number of quark flavors [42] as the renormalization group scale passes each physical quark mass. There is also a threshold at M_{SUSY}. Here the matching condition is the naive one of simple continuity due to the lack of knowledge about the superparticle spectrum. We take this scale to be variable to account for this ignorance.

We consider the simplest implementation of supersymmetry and run the couplings above M_{SUSY} to one loop. The superpotential for the supersymmetric theory is

$$W = \hat{\Phi}_u \hat{Q} \mathbf{Y}_u \hat{u}^c + \hat{\Phi}_d \hat{Q} \mathbf{Y}_d \hat{d}^c + \hat{\mathbf{Y}}_e \hat{C} \mathbf{Y}_e \hat{e}^c + \mu \hat{\Phi}_d \hat{\Phi}_u \,, \tag{3.2.1}$$

where the hat denotes a chiral supermultiplet. We assume the MSSM above M_{SUSY}, and a model with a single light Higgs scalar below it. This is done by integrating out one linear combination of the two doublets at M_{SUSY}, thereby

leaving the orthogonal combination in the standard model regime as the "Higgs doublet"

$$\Phi_{(SM)} = \Phi_d \cos\beta + \Phi_u \sin\beta \ , \qquad (3\,2\,2)$$

where $\bar{\Phi} = i\tau_2\Phi^*$, and where $\tan\beta$ is also the ratio of the two vacuum expectation values (v_u/v_d) in the limit under consideration. This sets boundary conditions on the Yukawa couplings at M_{SUSY}. Furthermore, in this approximation the quartic self coupling of the surviving Higgs at the SUSY scale is given by

$$\lambda(M_{SUSY}) = \frac{1}{4}(g_1^2 + g_2^2)\cos^2(2\beta) \qquad (3\,2\,3)$$

This correlates the mixing angle with the quartic coupling and thereby gives a value for the physical Higgs mass, M_{Higgs}. Using the experimental limits on the M_{Higgs} further constrains some of the results. By using the renormalization group we take into account radiative corrections to the light Higgs mass [52] and hence relax the tree level upper bound $M_{Higgs} \sim M_Z$ [53].

3.3 Bounds

We determine the bounds on M_t and M_{Higgs} by probing their dependence on β. In SUSY SU(5) $\tan\beta$ is constrained to be larger than one in the one light Higgs limit. It seems natural to us to require that $y_t \geq y_b$ up to the unification scale [54], thereby yielding an upper bound on $\tan\beta$. The initial values at M_Z for the gauge couplings are taken to be [14,55]

$$\begin{aligned} a_1 &= 0.016887 \pm 0.000040 \ , \\ a_2 &= 0.03322 \pm 0.00025 \ , \\ a_3 &= 0.100^{+0.004}_{-0.005} \end{aligned} \qquad (3\,3\,1)$$

where GUT normalization for a_1 is used. We use the set of four quark running masses defined at 1 GeV in the Particle Data book [29]: $m_u = 5.6$ MeV

$m_d = 9.9$ MeV, $m_s = 199$ MeV and $m_c = 1.35$ GeV. For the bottom mass we use the Gasser and Leutwyler bottom mass value of 5.3 GeV at 1 GeV which translates into a physical mass of $M_b = 4.6$ GeV [56]. To probe the dependence of our results on M_b we also study the case of $M_b = 5$ GeV, the typical value obtained from potential model fits for bottom quark bound states [57]. We also investigate the effect of varying M_{SUSY}. Given the values of the gauge couplings, we find unification up to a SUSY scale of 9.9 TeV, and as low as M_W. For empirical reasons we did not investigate solutions below that scale.

From Fig. 4 we determine that the lower end scale, M^L_{GUT}, of the unification region corresponds to an α_3 value of 0.104 at M_Z, while the higher end scale, M^H_{GUT}, corresponds to a value of 0.108 at M_Z for α_3. We find that the unification region is insensitive to the range of top, bottom, and Higgs masses considered. In our analysis of the bounds for M_t, the values for α_1 and α_2 are chosen to be the central values since their associated experimental uncertainties are less significant than for α_3. Demanding that y_b and y_τ cross at M^L_{GUT} and taking $\alpha_3 = 0.104$ then sets a lower bound on M_t. Correspondingly, demanding that y_b and y_τ cross at M^H_{GUT} and taking $\alpha_3 = 0.108$ yields an upper bound on M_t. These bounds are found for each possible value of β.

Fig. 5 shows the upper and lower bound curves for both M_t and M_{Higgs} as a function of β and for $M_{SUSY} = 1$ TeV and $M_b = 4.6$ GeV. When applicable we use the current experimental limit of 38 GeV on the light supersymmetric neutral Higgs mass [58] to determine the lowest possible M_t value consistent with the model. We find $139 \leq M_t \leq 194$ GeV and $44 \leq M_{Higgs} \leq 120$ GeV. We investigated the sensitivity of these results on M_{SUSY} in the range 1.0 ± 0.5 TeV. We find that the bounds on M_t are not modified, but the upper

bound on the Higgs is changed to 125 GeV, and the lower bound drops below the experimental lower bound

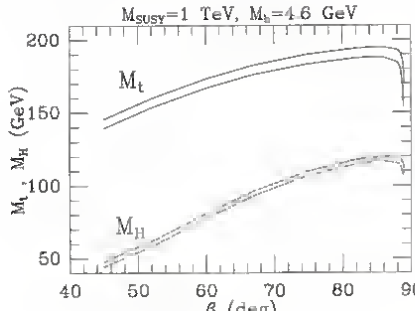


Figure 5 Plot of the top quark mass M_t and of the Higgs mass M_{Higgs} as a function of the mixing angle β for the highest value of α_3 (high curves) and the lowest value of α_3 (low curves) consistent with unification as per Fig 4

For $M_b = 5.0$ GeV we see an overall decrease in the top and Higgs mass bounds. $116 \leq M_t \leq 181$ GeV $M_{Higgs} \leq 111$ GeV Varying M_{SUSY} as above modifies the respective bounds The top mass lower and upper bounds become 113 and 119 GeV, respectively The upper bound on M_{Higgs} changes to 115 GeV We display the results of our analysis for the extreme case, $M_{SUSY} = 5.0$ TeV, in Fig 6 with $M_b = 4.6$ GeV

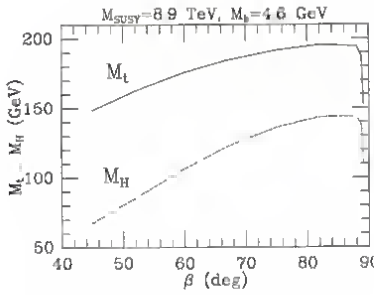


Figure 6 Same as Fig 5 for $M_{SUSY} = 8.9$ TeV and $M_b = 4.6$ GeV

This only significantly changes the upper bound on $M_{H^{SM}}$ to 144 GeV compared to the $M_{SUSY} = 1$ TeV case

We have also run y_t up to the unification region and compared it with y_b and y_τ to see what the angle β must be for these three couplings to meet [59], as in an SO(10) or E_6 model [10 11] with a minimal Higgs structure. It is clear that this angle is precisely our upper bound on β as described earlier In Fig 7 we display y_t/y_b at the GUT scale as a function of $\tan\beta$ for $M_{SUSY} = 1$ TeV and for the two bottom masses we have considered

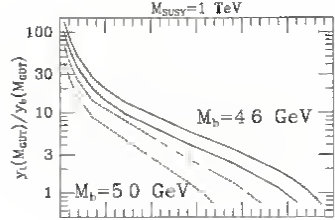


Figure 7 Plot of the ratio of the top to bottom Yukawas y_t/y_b for two different bottom masses (solid and dashed curves) as a function of $\tan\beta$ for the highest value of α_3 (high curves) and the lowest value of α_3 (low curves) consistent with unification as per Fig 4

If we demand that the ratio be one we can determine the mixing angles for the low and high ends of the unification region Then going back to Fig 5 we find as expected a much tighter bound on the masses of the top and of the Higgs Indeed, for $M_b = 4\,6$ GeV we have $49\,40 \leq \tan\beta \leq 54\,98$, which yields $162 \leq M_t \leq 176$ GeV and $106 \leq M_{Higgs} \leq 111$ GeV When $M_b = 5\,0$ GeV we obtain $31\,23 \leq \tan\beta \leq 41\,15$, which gives $116 \leq M_t \leq 147$ GeV and $93 \leq M_{Higgs} \leq 101$ GeV

Several issues have been left untouched We have not implemented the supersymmetric two loop beta functions and the corresponding thresholds The effects of soft SUSY breaking terms were not investigated nor was the possible role of a large top mass on this breaking Also, we have integrated out

all the supersymmetric particles at the same scale. It would be interesting to study the effect of lifting this restriction. Such a study is underway

CHAPTER 4
MASS AND MIXING ANGLE PATTERNS IN THE SM

4.1 Models of the Yukawa Sector

Many models have been proposed to explain the peculiar structure of the Yukawa couplings. In a certain basis, these Yukawa matrices are well approximated by the matrix [60]

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.1.1}$$

which expresses the fact that one family is so much heavier than the other two. The theoretical temptation has been to express the masses of the lighter families and their mixings as radiative effects [61]. Yet this approach has not yielded any satisfactory models. Another has been to use as the starting point the "democratic" matrix where all entries are equal, thereby producing two zero eigenvalues. This type of idea could be implemented by condensates formed by new flavor blind strong forces [62]. Yet another approach is to think of the "elementary" particles as composites and use the Yukawa matrices as hints to infer their structure [63]. In the following, however, we concentrate on Yukawa matrices that arise naturally in the context of Grand Unified Theories.

In the context of the SU(5) GUT [9], several mass relations arise, based on simple assumptions for the possible Higgs structure. The mass term for the down quarks and leptons comes from the Yukawa interaction of the $\overline{5}$ and 10 of fermions. This leads to relations between the charge $-1/3$ quarks' and

the charged leptons' Yukawa couplings. With only a $\overline{5}$ of Higgs, one obtains equality between the τ lepton and bottom quark masses at the GUT scale

$$m_b = m_\tau \ . \tag{4.1.2}$$

To the level of approximation used at the time, this relation was found to be consistent at experimental scales, after taking into account the running of the quark masses [64]. Similar relations apply to the lighter two families, but are clearly incompatible with experiment. To alleviate this, a new scheme was proposed [65] with a slightly more complicated Higgs structure (using a **45** representation in conjunction with the $\overline{5}$). It replaces the above with the more complicated relations for the two lighter families

$$\begin{aligned} m_d &= 3m_e \\ 3m_s &= m_\mu \end{aligned} \tag{4.1.3}$$

The situation concerning the mixing angles is equally intriguing. There happens to be a near numerical equality between the square of the tangent of the Cabibbo angle and the ratio of the down to the strange quark masses (determined from current algebra). This Gatto-Sartori-Tonin Oakes (GSTO) relation [66] reads

$$\tan^2\theta_C \approx \sqrt{\frac{m_d}{m_s}} \tag{4.1.4}$$

It has provided the central inspiration in the search for Yukawa matrices. Very general classes of matrices with judiciously chosen textures [67] (i.e., zeroes in the right places) could reproduce this relation at least approximately.

In the context of $SO(10)$ [10], these three different relations could all be obtained in one model [68] with the required texture enforced naturally by discrete symmetries at the GUT scale. In this model the mixing of the third family with the two lighter ones is dictated exclusively by the the charge 2/3

quarks' Yukawa matrix. There ensues an (GSTO)-like relation for the mixing of the second and third families [68]

$$V_{cb} = \sqrt{\frac{m_c}{m_t}}\,, \tag{4.1.5}$$

which provides a relation between the top quark mass and the lifetime of the B meson.

These four relations can all be obtained if one takes the Yukawa mixing matrices to be of the form [65 68] (shown here in a specific basis)

$$\mathbf{Y}_u = \begin{pmatrix} 0 & P & 0 \\ P & 0 & Q \\ 0 & Q & Y \end{pmatrix}\,,$$

$$\mathbf{Y}_d = \begin{pmatrix} 0 & R & 0 \\ R & S & 0 \\ 0 & 0 & T \end{pmatrix} \qquad \mathbf{Y}_e = \begin{pmatrix} 0 & R & 0 \\ R & -3S & 0 \\ 0 & 0 & T \end{pmatrix} \tag{4.1.6}$$

This form has been recently rediscovered by several groups [69 70].

Although derived with specific and sometimes complicated Higgs structures in mind (as in the $SO(10)$ model), these relations may well prove sturdier than the theories which generated them. In this chapter we examine the relations in the context of the standard model at varying scales all the way to Planck scale. We then extend, in the next chapter, the analysis to the minimal supersymmetric extension of the standard model and compare the effect of this extension on their compatibility at some unified scale.

4.2 The Renormalization Group

We first use experiment to fix the parameters of the standard model at lower energies. We then use these values as initial conditions in the renormalisation group running to shorter length scales, using the $\overline{\rm MS}$ scheme [17]. In the standard model, we use two loop renormalization group equations in

evolving the couplings In the supersymmetric extension, we work to one-loop [21] In each case, we include a proper treatment of thresholds and make no approximations in the Yukawa sector Our incomplete knowledge of the standard model parameters forces us to repeat the analysis for a range of allowed values of the top quark and Higgs masses Although the top quark mass is not exactly known it is believed to lie somewhere between 91 GeV [71] and 200 GeV, the lower limit being set by direct experimental searches, the upper one by the radiative effect of the top quark mass on the ratio of neutral to charged current processes (ρ-parameter) In these runs, we take $g_3(M_Z) = 1\ 191$ and the physical bottom quark mass $M_b = 4\ 89$ GeV

Let us summarize the salient features of the renormalization group running in the standard model. At the one loop level the gauge couplings are unaffected by the other couplings in the theory On the other hand, the Yukawa couplings are affected at one loop by both the gauge and Yukawa couplings Since the top Yukawa coupling is at least as big as the gauge couplings at low energy, that means the running of the Yukawas is sensitive to mostly the top Yukawa and the QCD gauge couplings Thus we can expect the mass and mixing relations we have just described to be sensitive to the value of the top quark mass. The Higgs quartic self-coupling enters in the running of the other couplings only at the two loop level, so that its effect on the quark and lepton parameters is small However, its own running is very sensitive to the top quark mass it can become negative as easily as it can blow up correspondingly to vacuum instability or to strong self interaction of the Higgs (triviality bound), respectively The discovery of the Higgs with mass outside these bounds would be a signal for physics beyond the standard model. The graphs in Fig 8 summarize these bounds for representative values of the top quark mass

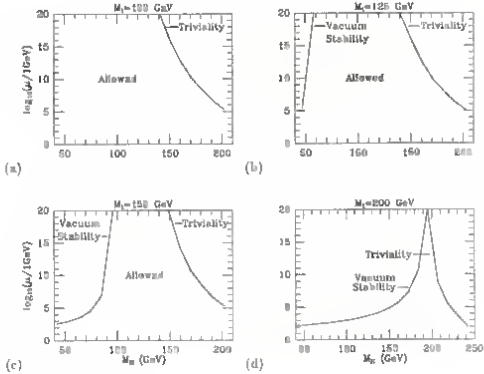


Figure 8 Vacuum stability and triviality bounds on the Higgs mass for (a) M_t=100 GeV (b) M_t=125 GeV (c) M_t=150 GeV and (d) M_t=200 GeV giving scales of expected new physics beyond the standard model

For example, if $M_t = 150$ GeV, we see from the corresponding plot that a Higgs mass between 95 and 150 GeV need not imply any new physics up to Planck scale. However, if the Higgs were observed outside of this range, then some new physics must appear at the scale indicated by the curve, either because of vacuum instability if $M_H < 95$ GeV or because the Higgs interaction becomes too strong if $M_H > 150$ GeV. It is amusing to note that it is for comparable values of the top and Higgs masses that these bounds are least restrictive, but it is important to emphasize that a high value of the top with a relatively low

value of the Higgs necessarily indicates the presence of new physics within reach of the SSC. Subsequently, when examining the mass and mixing angle relations in the context of the standard model, we will make the choices for M_t and M_H in our renormalization group runs consistent with these bounds. For a chosen value of M_t varying M_H within the vacuum stability and triviality bounds does not affect any of our results, and we will therefore choose a corresponding representative value of M_H.

As is well known, the standard model shows no apparent inconsistencies until perhaps the Planck scale where quantum gravity enters the picture. The nature of the physics to be found between our scale and the Planck scale is a matter of theoretical taste. At one extreme, because of the values of the gauge couplings, new phenomena may be inferred every two orders of magnitude. At the other, there is the possible desert suggested by GUTs, however, the absence of new phenomena over many orders of magnitude cannot be understood (perturbatively) unless one generalizes the standard model in some way to solve the hierarchy problem. Supersymmetrizing the standard model at an experimentally accessible scale can accomplish this. This particular scenario is bolstered by the fact that with such "low energy" supersymmetry, the three gauge couplings of the standard model meet at one scale ($\sim 10^{16}$ GeV) at the perturbative value of $\sim 1/26$ [14]. The collapse of the GUT triangle in the supersymmetric extension fixes two scales: the one at which the gauge couplings unify, the other at the threshold of supersymmetry. Minimal supersymmetry implies two Higgs doublets and eliminates the feisty quartic self-coupling of the standard model. But there appears an extra parameter, the ratio of the vacuum values of these two doublets parametrized by an angle β, $\tan\beta = v_u/v_d$, where v_u (v_d) is the vacuum expectation value of the Higgs field that gives

mass to the charge 2/3 (−1/3 −1) fermions. In the following, we examine certain relations among masses and mixing angles in the context of the standard model itself and in its minimal supersymmetric extension. In the latter, we only treat the case of one light Higgs.

4.3 Relation (I) $m_b = m_\tau$

This relation is the most natural one in the $SU(5)$ theory, and it could be expected to be valid at scales where the standard model gauge couplings are the closest to one another. We examine its validity for three different physical values of the top and Higgs masses in the standard model. The results are summarized in Fig. 9.

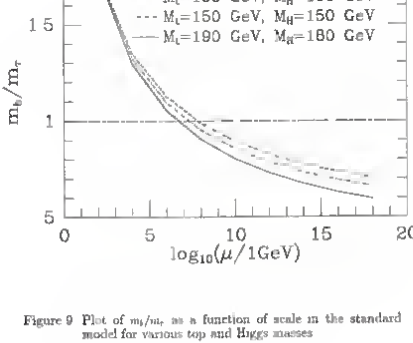


Figure 9. Plot of m_b/m_τ as a function of scale in the standard model for various top and Higgs masses.

The noteworthy feature of the figure is that this simplest of the SU(5) relations is valid at an energy scale many orders of magnitude removed from that at which the gauge couplings tend to converge [21] Our result is vastly different from that of the original investigations in Ref 64 We have improved on their work by including two loop effects in the running of the quark Yukawas, by taking into account the full Yukawa sector and most importantly by incorporating QCD corrections in the extraction of the bottom quark mass

4.4 Relations (II), $m_d = 3m_e$, $3m_s = m_\mu$

We now turn to the more complicated relations among masses of the two lighter families There are large theoretical uncertainties in the extraction of the masses of the three lightest quarks from experiment although the mass ratios are known more accurately Following Refs. 56 we take their values to be $m_d/m_u = 1.8$ and $m_s/m_d = 21$ so that specifying m_s fixes m_d and m_u We note that m_s/m_d and m_μ/m_e effectively do not run [17] Therefore given this value for m_s/m_d we do not expect relations (II) to be both satisfied exactly since $m_\mu/3m_e \approx 23$ The uncertainties in the light quark masses are accounted for by examining the ratios $m_d/3m_e$ and $3m_s/m_\mu$ for a range of $m_s(1\ \mathrm{GeV})$ values from 140 to 250 MeV We have run these same ratios for representative values of the top and Higgs masses but find the results to be fairly insensitive to the value of the top Therefore in Fig 10 we only present results for top and Higgs masses of 190 GeV and 180 GeV, respectively

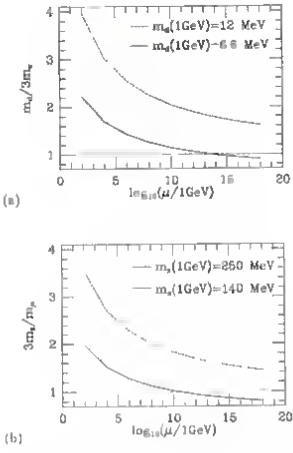


Figure 10 Plots of (a) $m_d/3m_e$ and (b) $3m_s/m_\mu$ as a function of scale in the standard model for (a) M_t=190 GeV and (b) M_H=180 GeV

Unlike relation (I) which holds only at $\sim 10^7$ GeV, we see that relations (II) can hold within $\sim 5\%$ at 10^{16} GeV for acceptable values of the light quark masses

4.5 Relation (III) $\tan\theta_c = \sqrt{m_d/m_s}$

We find the GSTO relation to be quite independent of scale The reason is that the Cabibbo angle effectively does not run as noted earlier and the ratio of light quarks is essentially unaffected by QCD, since both are far away from the Pendleton-Ross [72] infrared fixed point Further, we observe that their numerical values are fairly independent of the value of the top quark mass and of the Higgs mass The agreement is spectacular, hovering around the 4% level For example for $M_t = 100$ GeV and $M_H = 100$ GeV, we find that $\tan\theta_c/\sqrt{m_d/m_s} = 1.038$ from M_Z to Planck scale

4.6 Relation (IV) $V_{cb} = \sqrt{m_c/m_t}$

This expression involves the top quark mass directly, which may thus be predicted from this relation On the other side of the equation the experimental value of the "23" element of the CKM matrix, V_{cb}, is known only to within $\sim 10\%$, $V_{cb} = 0.043 \pm 0.006$ The value of V_{cb} at all scales is obtained by running the CKM angles These numerical results do not depend on the value of the CP violating phase

We note that because of the Pendleton-Ross fixed point, the ratio of the two quark masses runs appreciably in the infrared region We find that for a top quark in its lower allowed range, $91 \sim 150$ GeV, this relation fails over all scales Accordingly we present our results for values of V_{cb} and M_t for which the relation can be satisfied below Planck scale. We use for V_{cb} values ranging from its central value of .043 to .050 Here and in the following we take the value of m_c at 1 GeV to be 1.41 GeV

The results of our runs can be summarized in Fig 11, in which we plot both V_{cb} and $\sqrt{m_c/m_t}$ as a function of scale

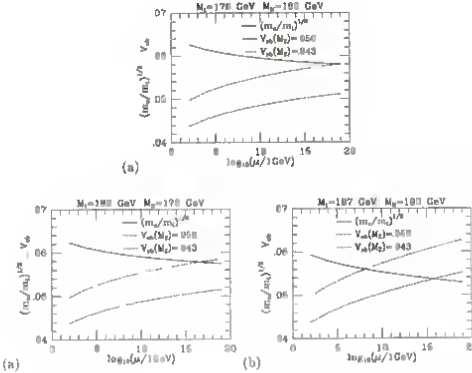


Figure 11 Plot of V_{cb} and $\sqrt{m_c/m_t}$ as a function of scale in the standard model for (a) M_t=178 GeV (b) M_t=180 GeV and (b) M_t=187 GeV for both the central value (043) and the maximal value (050) of $V_{cb}(M_Z)$

From the first plot we see that the top quark has to be at least 178 GeV for $\sqrt{m_c/m_t}$ to meet V_{cb} at the Planck scale The second plot shows that M_t = 180 GeV allows for this relation to be easily satisfied at 10^{16} GeV if $V_{cb}(M_Z)$ = 05 This means that a few GeV difference in M_t changes the meeting of the curves (both of which are affected by M_t) by three orders of magnitude! Finally, in the third plot we see that for this relation to be valid at the unification scale using the central value of V_{cb}, a 187 GeV top quark

is needed We conclude that given the uncertainties in the value of V_{cb}, this relation may well be valid as long as $M_t > 175$ GeV

4.7 Other Possible Mass Relations

Another interesting mass relation involves the ratio of the determinants of the charge 1/3 to charge 1 mass matrices and should equal one if relations (I) and (II) are valid We note that independent of the top mass (91–200 GeV) this weaker (less predictive) relation ($m_d m_s m_b = m_e m_\mu m_\tau$) can be satisfied at 10^{16} GeV for quark masses within the range stated above

The relations considered so far have been motivated by specific theoretical models There are other relations which are not similarly motivated but which may nevertheless hint at an underlying unified structure In our search for simple relations among the quark masses we considered an appealing "geometric mean" relation in the up sector namely

$$m_u m_t = m_c^2 \tag{4 7 1}$$

This relation favors higher top quark masses and can be satisfied well at 10^{16} GeV in the $M_t = 100$ GeV $M_H = 180$ GeV scenario with an up quark mass compatible with that needed to satisfy relations (II) at 10^{16} GeV

A similar relation involving the down type quarks was tested

$$m_d m_b = m_s^2 \tag{4 7 2}$$

This relation favors higher top quark masses as well In fact, in order to satisfy Eqs (4 7 1) and (4 7 2) at 10^{16} GeV as well as relations (II) given a fixed value for m_s(1 GeV) within the range cited, the top quark mass would have to be larger than 200 GeV Fortuitously, such a value would also favor relation (IV)

These geometric mean relations have been discussed in the literature recently [73].

Relation (4.7.1), in fact, can be given a theoretical motivation. It can be derived from a different texture for the charge 2/3 Yukawa matrix. The following texture, introduced by Giudice [74], we will show lead to the geometric mean relation in the up-sector

$$Y_u = \begin{pmatrix} 0 & 0 & b \\ 0 & b & 0 \\ b & 0 & a \end{pmatrix} \tag{4.7.3}$$

Indeed, consider the mass matrix in the up sector obtained after diagonalization, it is proportional to

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & -m_t \end{pmatrix} \tag{4.7.4}$$

where, since Y_u is symmetric, requires simply a unitary transformation as opposed to a bi-unitary one as per the general case in order to be diagonalized. This means that the matrices in Eqs. (4.7.3) and (4.7.4) have the same invariants. In the case of a 3×3 matrix they are the trace, S_2, and the determinant. The trace and the determinant are well known: what we called S_2 is the second order symmetric polynomial in the eigenvalues (the trace and the determinant are respectively the corresponding first and third order polynomials). For Y_u and the mass matrix (up to the $\sqrt{2}/v$ proportionality factor), they are

$$\begin{cases} \text{Trace} & b+a &=& m_u+m_c-m_t \\ S_2 & ba-b^2 &=& m_u m_c - m_c m_t - m_t m_u \\ \text{Det} & -b^3 &=& -m_u m_c m_t \end{cases} \tag{4.7.5}$$

Now it is clear from the determinant equation above that the choice $b=m_c$ gives us the geometric mean, $m_c^2 = m_u m_t$. However, we need to check that this choice is consistent with the other two invariants. Replacing $b=m_c$ in

the equation for S_2 gives that $a = m_u - m_t$. Then adding a and b we recover the trace equation. It does follow therefore that the up-sector geometric mean relation follow from the Gudice texture. This was just done to motivate the relation but testing the texture itself under the renormalization group has been studied by several authors [74,75]. We have not found a similar motivation for the geometric mean relation in the charge $1/3$ sector.

To conclude our analysis of the standard model case, we see that it is hard to arrive at a unified picture. The scale at which relation (I) tends to be satisfied does not coincide with that at which the others are valid. Still the disagreement is never too large, which makes us hope that small course corrections in the running of the parameters allow most if not all of these relations to hold simultaneously at a unified scale. It is remarkable that for a top quark at the upper reaches of its allowed range, the long life of the bottom quark lends plausibility to the $SO(10)$ inspired relation (IV).

CHAPTER 5
MASS AND MIXING ANGLE PATTERNS IN THE MSSM

5.1 Relations (I)-(IV)

We have shown in chapter three that low energy supersymmetry allows relation (I) to be valid at the scale of gauge unification. In fact, if this relation is imposed at the unification scale, the top quark mass is fixed (up to experimental errors in g_3), for a given value of the angle β, as displayed in Fig. 12 for two different scales of supersymmetry breaking.

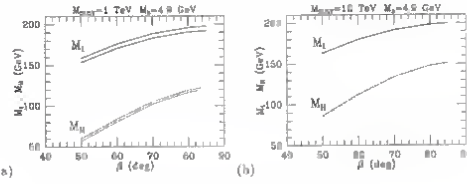


Figure 12 Plot of M_t and M_B as a function of β. For (a) M_{SUSY} = 1 TeV the high curves (low curves) correspond to the highest (lowest) value of $\alpha_3(M_Z)$ consistent with unification. For (b) M_{SUSY}=10 TeV the low and high curves coincide.

The reason is that the bottom Yukawa running depends both on g_3 and on the top Yukawa. We note that unlike the standard model case the top Yukawa starts increasing at shorter length scales, providing us with an upper limit on the mass of the top quark. In addition, by relating the scalar quartic self coupling below the supersymmetry breaking scale, to β, g_1, and g_2 above it the mass of the lightest Higgs is fixed in terms of the mass of the top (or equivalently β). In the following, we shall take relation (1) as valid at M_{GUT} and assume the correspondence between β, M_t and M_H depicted in these figures. We note, therefore, that this limits the top mass we can consider to be $\lesssim 200$ GeV. For $M_{SUSY} = 1$ TeV, the gauge unification occurs between a low of 6.92×10^{15} GeV (Low M_{GUT}) and a high of 1.26×10^{16} GeV (High M_{GUT}) corresponding to $g_3(M_Z) = 1.171$ and $g_3(M_Z) = 1.197$, respectively. The error bars in the strong coupling allow for a SUSY scale as high as ~ 10 TeV, with unification at 6.46×10^{15} GeV.

We need now to exploit the relations (II,IV) to constrain M_t and therefore β and M_H. For $M_{SUSY} = 1$ TeV we treat two cases: the first where unification takes place at its lowest value (Low M_{GUT}) and the second where it is at its highest value (High M_{GUT}). In the following we will not discuss our results for the supersymmetry breaking scale of 10 TeV, since it adds nothing to our conclusions.

In the case of the mass relations among the light quark and lepton masses (relations (II)), we find that our plots do not depend on M_t, therefore we only display them for a representative value. Here we do not follow the strategy used in the standard model case (i.e., we do not vary m_s), although we still keep the ratios m_d/m_u and m_s/m_d fixed. Instead, we look for that value of m_s(1 GeV) which gives us the best agreement for relations (II) both for the

low and high M_{GUT} (we cannot expect exact agreement at one scale, since m_s/m_d does not run) For instance, we can get the same value of m_s(1 GeV) in two different ways, either by demanding that at low M_{GUT} $m_d/3m_e = 1\,1$ and $3m_s/m_\mu = 1$ or at high M_{GUT} that $m_d/3m_e = 1$ and $3m_s/m_\mu = 9$ In both cases, the masses of the lightest quarks at 1 GeV are $m_u = 3\,80$ MeV $m_d = 6\,67$ MeV, and $m_s = 141$ MeV These results are summarized in Fig 13,

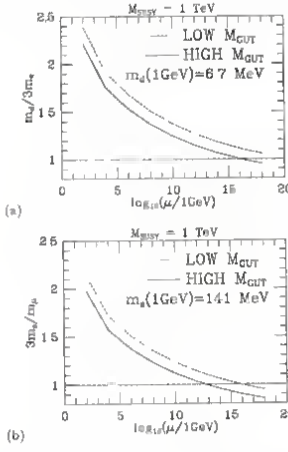


Figure 13 Plots of (a) $m_d/3m_e$ and (b) $3m_s/m_\mu$ as a function of scale in the SUSY case with M_{SUSY}=1 TeV, for the low and high unification scales (no appreciable dependence on M_t was found)

In the above, our philosophy has been to take the known low energy data and using the renormalization group, derive its implications at high energy. As we did in chapter three for relation (I), we could impose both of relations (II) at one unification scale (low or high M_{GUT}). This would fix m_s/m_d at this scale, and since m_s/m_d and m_μ/m_e do not run, that would yield $m_s/m_d = 23.6$, a value that is only 12% larger than the value in Refs. 56. Furthermore, the results of our runs yield the masses at 1 GeV of the down and strange quarks to be $m_d = 5.56$ MeV, $m_s = 138$ MeV, in the low M_{GUT} case, and $m_d = 6.49$ MeV, $m_s = 153$ MeV, in the high M_{GUT} case. We note that this approach has also been taken by the authors of Ref. 70.

Before discussing relation (IV), let us note that, with supersymmetry, relation (III) is again well satisfied at all scales. We now turn to relation (IV).

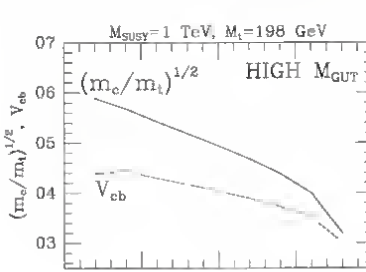


Figure 14. Plot of V_{cb} and $\sqrt{m_c/m_t}$ as a function of scale in the SUSY case with M_{SUSY}=1 TeV for M_t=198 GeV and for $V_{cb}(M_Z)$= 043.

As we did in the standard model case we display our results both for the central value of V_{cb} (043) and for its upper value (050) Then we look for values of M_t which give us agreement at the unification scale (low or high M_{GUT}) Using the central value for V_{cb}, we find no agreement at the unification scale. However, this relation is satisfied at Planck scale, if we use both a high M_{GUT} and $M_t = 198$ GeV (the highest possible value consistent with relation (1)) as displayed in Fig 14

We have also made several runs with a higher value V_{cb} There, the relation can actually be satisfied provided that we use the high M_{GUT} scale and $M_t = 198$ GeV as shown in Fig. 15.

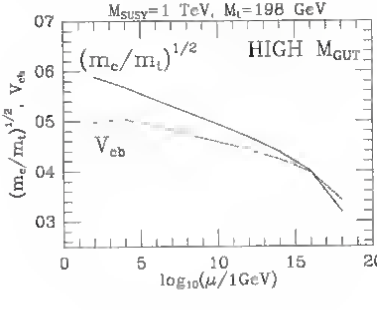


Figure 15 Same as Fig 14 with $V_{cb}(M_Z)$= 050 and M_t=198 GeV

In the low M_{GUT} case, the two curves meet closer to the Planck scale. In fact, theory does not dictate to us the exact scale at which the $SO(10)$ inspired relation is valid, it could be much higher than the scale of unification of the standard model's gauge couplings. To account for this, we now plot, in Fig. 16 V_{cb} as a function of M_t assuming that relation (IV) is valid at M_{GUT}, $10M_{GUT}$, and $100M_{GUT}$, and using the higher value of $g_3(M_Z)$.

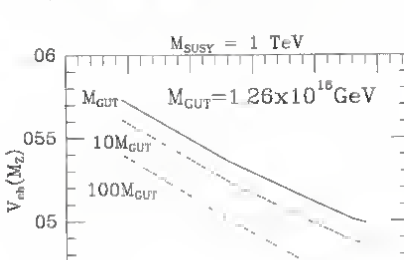


Figure 16 Plot of $V_{cb}(M_Z)$ as a function of M_t assuming relation (IV) holds at various scales $\geq M_{GUT}$

Given an initial value of V_{cb} at M_Z, Fig. 16 can be used to determine the needed M_t (and hence β) to satisfy relation (IV) at M_{GUT}, $10M_{GUT}$, or $100M_{GUT}$. We can see from this figure that as long as V_{cb} is larger than its central value

then relation (IV) can be satisfied above the $SU(5)$ GUT scale and still allow for a lower value of M_t

5.2 Other Possible Mass Relations

The relation in Sec 4 7 involving the determinants of the charge $-1/3$ and charge -1 fermion mass matrices holds in the minimal supersymmetric model at 10^{14} GeV in the high M_{GUT} case and at 10^{16} GeV in the low M_{GUT} case. In both cases, this relation holds within $\sim 10\%$ at 10^{16} GeV These results are true for the light quark masses chosen in Sec 5 1 and for all $160 \leq M_t \leq 198$ GeV

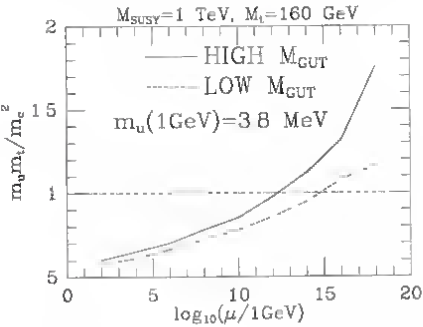


Figure 17 Plot of $m_u m_t/m_c^2$ as a function of scale for the highest value of $\alpha_3(M_Z)$ (high curve) and the lowest value of $\alpha_3(M_Z)$ (low curve) and for M_t=160 GeV

A priori one might naively assume that $m_u m_t = m_c^2$ could be easily satisfied because of the uncertainty in m_t. However two facts make the relation viable in the supersymmetric case. First, the value predicted for the top mass is within the range allowed by experiment and the ρ-parameter bound. Second, and most remarkable, is the fact that this top mass value is compatible with relations (I)-(III) given the choice of light quark masses in Sec. 5.1. In Fig. 17 we display the running of the ratio $m_u m_t/m_c^2$ for the low and high M_{GUT} cases. We show the curves representing the lower M_t value of 160 GeV for which the relation is best satisfied at M_{GUT}. This relation is incompatible with relation (IV) however, since the latter favors a higher top mass. We note that m_c affects these two relations in an "inverse" manner. A lower experimental value for the charm quark mass favors relation (IV) whereas a higher experimental value favors the geometric mean relation.

One may also consider the geometric mean relation in the down sector. We find however, that this relation fails to hold in the supersymmetric case. Other relations among the Yukawa couplings have been considered in the literature. Theoretical bias or numerology can lead to still other relations valid at some unifying scale. In all cases a thorough renormalization group analysis will be required in investigations of a possible deeper structure.

At this point summarizing we can say that in the standard model case there are many unsatisfactory features, not the least of which is the failure of the gauge couplings to unify within experimental error, forming a GUT triangle. The simplest of the $SU(5)$ relations $m_b = m_\tau$, can only be satisfied some ten orders of magnitude from the scale of the GUT triangle. The other relations $m_d = 3m_e$, $3m_s = m_\mu$, $\tan\theta_c = \sqrt{m_d/m_s}$, and $V_{cb} = \sqrt{m_c/m_t}$, could be satisfied at 10^{15} GeV. The geometric mean relations we considered

can also be simultaneously satisfied, but this requires a top quark mass greater than 200 GeV

In the SUSY case for which the GUT triangle collapses, we achieved a striking agreement for the four main relations considered. But, for this to be true, several things must occur first V_{cb} must be larger than its presently measured value, second the top quark mass must be around 190 GeV (if it is a bit lighter then agreement dictates that V_{cb} should be larger still), third the Higgs mass should hover around 120 GeV These conclusions are qualitatively correct if one demands maximum agreement An analysis which recently appeared in the literature has reached similar conclusions [70] However, it is difficult to arrive at more definite numbers without an exhaustive analysis of the parameter space

CHAPTER 6
LIE ALGEBRAS AND LAX OPERATORS

Several approaches to constructing the Lax operators have been developed. The matrix procedure discussed by Drinfel'd and Sokolov [23] defines first a matrix eigenvalue equation. The system incorporates knowledge of the Cartan subalgebra and root system of some embedding affine Lie algebra $\mathcal{G}$. Starting with an affine Lie algebra facilitates the construction of an integrable system from the resulting Lax pair operators. To fix the gauge invariance in the matrix system, the gradation conventions of Drinfel'd and Sokolov require that one of the simple roots, say the m^{th}, must be removed from the affine system. The resulting system is denoted by $(\mathcal{G}\ c_m)$. This is equivalent to deleting the m^{th} Dynkin vertex. For the most part, Drinfel'd and Sokolov choose the "canonical" gauge in which to express the coordinate dependent terms. In this gauge, Lax operators generate the regular KdV hierarchy equations.

The modified KdV(mKdV) equations can be generated by expressing the coordinate term $q(x)$ in the "diagonal" gauge. The canonical Lax operators can then be recovered using the well known Miura transformations. The diagonal gauge is technically simpler than the canonical gauge. Furthermore, the final Lax operator is in a factorized form which has been used to quantize the theory [76].

Our focus will be on the explicit construction of the Lax (pseudo)differential operators in the diagonal gauge using a simple diagrammatic technique [24]. In most cases this technique arrives at these operators much quicker than a direct

application of the scheme of Drinfel'd and Sokolov. Furthermore, the scheme also applies to higher representations of the embedding affine Lie algebra.

As noted above, each Lax operator can be associated with a representation of some affine Lie algebra. Thus we give a brief review for constructing weight diagrams corresponding to these representations.

Recall [77], one can associate uniquely to every irreducible representation of a basic Lie algebra a highest weight vector. For each highest weight one can construct a weight diagram which encodes all relevant information concerning the particular representation studied, e.g., from it one can build explicit matrix representations of the generators of the Cartan subalgebra as well as the various raising and lowering operators. The level of a weight is the number of lowering operators applied to the highest weight which produces that weight. Finally, the height γ of the weight diagram is the level of the lowest weight.

Weight diagrams are generated by subtracting rows of the Cartan matrix initially from the highest weight vector written in the Dynkin basis. Rules of construction can be summed up as follows:

1. Subtract the i^{th} row of the Cartan matrix n times from a weight vector whose i^{th} component has a positive value n.
2. When weight vectors have more than one positive component, subtract all possible permutations of the appropriate Cartan rows.

A theorem due to Dynkin [78] states that the final weight diagram is always "spindle shaped". In other words: a) the number of weight vectors at the level k is equal to the number at level $\gamma - k$; b) the number of weights at level $k+1$ is greater than or equal to the number at level k for $k < \frac{\gamma}{2}$.

For an explicit example consider the algebra A_2. Though this is almost a trivial case, the results will be useful for the next section. The Dynkin diagram is given by

where, recall, the single bar represents 120°. The Cartan matrix is then easily found to be

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{6.1}$$

The highest weight vector of the fundamental representation is (10) [79]. Since a positive one appears in the first place, we subtract the first row of the Cartan matrix one time. This gives the weight (-11). Now, due to the one in the second place, we subtract the second row of the Cartan matrix once to get $(0-1)$. This completes the process since no positive components remain. The result is the height two weight diagram

$$\begin{array}{c} (10)_1 \\ \downarrow 1 \\ (-11)_2 \\ \downarrow 1 \\ (0-1)_3 \end{array} \tag{6.2}$$

where the subscripts on the weight vectors indicate a counting of the vectors. The ones adjacent to the arrows represent the normalization factors of the corresponding negative simple roots. These values are fixed by the commutation relations of the Lie algebra. To simplify our diagrams, we will not display values of unity. Later, we will see that the procedure for building weight diagrams is slightly modified in the supersymmetric case.

For an affine Lie algebra [80] since there exists a linear combination among simple roots, weight diagrams of affine representations generally have infinite extent. However, some affine representations give cyclic weight diagrams of finite extent. In fact, it is these cyclic cases that are crucial to the scheme

below To produce a cyclic weight diagram, the affine component which is appended to the highest weight vector of the underlying non-affine Lie algebra, unlike the non affine weight components may have to be assigned a negative value

As an explicit example, consider the non twisted affine algebra $A_2^{(1)}$ To generate the cyclic weight diagram corresponding to the canonical representation, start with the weight vector (-110), where -1 corresponds to the affine root Fig 18 gives the Dynkin diagram and subsequent Cartan matrix which then generates the displayed resultant weight diagram

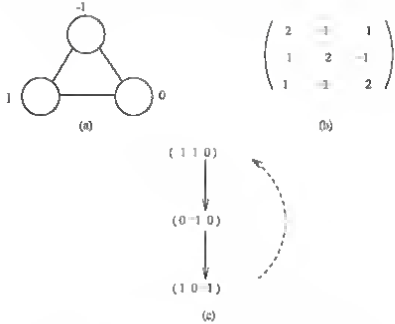


Figure 18. (a) Dynkin diagram (b) Cartan matrix and (c) cyclic weight diagram of the (-110) representation of $A_2^{(1)}$ The dashed arrow is the deleted root

This particular cyclic weight diagram can further be thought of as the affine extension of the highest weight diagram based on the fundamental representation of the basic Lie algebra A_2. This is easy to see by removing everywhere the component due to the affine root. However, this is not always the case

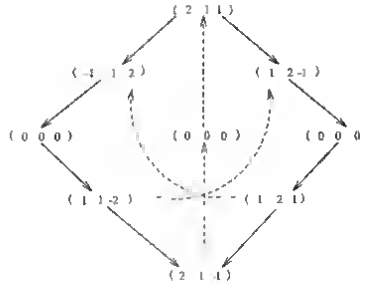


Figure 19 Cyclic weight diagram of $A_2^{(1)}$ from the weight (-211)
The dashed arrows are the deleted root

For example Fig. 19 displays the cyclic weight diagram constructed with the weight vector (-211) where now the affine component is -2

Although (11) generates the highest weight diagram of the adjoint representation of A_2, we see by comparing with Fig. 20 that the affine extension contains an extra zero weight (000)

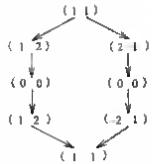


Figure 20 Highest weight diagram of the adjoint representation of A_2. (11)

In general, the affine component in the affinely extended vector associated with the highest weight vector of a basic Lie algebra, will always be negative However we will give an example below showing that some supersymmetric cases require positive affine components

CHAPTER 7
LAX OPERATOR CONSTRUCTION

7.1 Standard Construction of Lax Operators

The Lax operator [81] $L(x,t)$ is defined to be linear and Hermitian. Furthermore, it satisfies the characteristic equation

$$L(x,t)\phi(x,t) = \mu\phi(x,t) \tag{7.1.1}$$

where the eigenvalue μ is required to be constant under nonlinear evolution. In other words, the nonlinear behavior of the eigenfunctions $\phi(x,t)$ are governed by some operator $A(t)$, which may be nonlinear, via the equation

$$\frac{\partial\phi(x,t)}{\partial t} = A(t)\phi(x,t) \tag{7.1.2}$$

Furthermore, $A(t)$ enters into the differential scalar Lax equation

$$\frac{\partial L(x,t)}{\partial t} = [A(t),\ L(x,t)], \tag{7.1.3}$$

which generates the integrable KdV equations.

The matrix construction of Lax operators utilizing generators of some embedding affine Lie algebra $\hat{g}^{(k)}$ reviewed by Drinfel'd and Sokolov [23] begins with a matrix operator of the form

$$\mathcal{L} = I\frac{\partial}{\partial x} + A + g(x) \tag{7.1.4}$$

where I denotes the $N \times N$ dimensional unit matrix, and to simplify notation we have suppressed the argument t. The third term is discussed below. The

second term is generated by the negative simple roots E_i of the embedding affine Lie algebra. In the gradation conventions of Drinfel'd and Sokolov we have the circulant matrix

$$\Lambda = \sum_{i=0}^{r} c_i E_i \tag{7 1 5}$$

The procedure is then to reduce the system of linear equations given by the kernel matrix equation

$$\mathcal{L}\vec{\psi}(x) = 0 \tag{7 1 6}$$

where $\vec{\psi} = (\psi_1, \psi_2, \ldots \psi_N)$ to the linear differential eigenvalue equation, Eq (7 1.1) where the vacuum solution ϕ is a function of the components of the eigenfunction $\vec{\psi}$. Drinfel'd and Sokolov show that such reduction is possible if one removes a simple root, say the m^{th}, from the affine root system. They denote this situation $(\tilde{\mathcal{G}}^{(k)}, c_m)$, which is in the homogeneous or standard gradation [82]. The coefficients in (7 1 5) are then assigned the values $c_{i\neq m} = 1$ and $c_m = \lambda$, where λ is a constant function of the spectral parameter μ.

Removing an element from the simple root system is equivalent to deleting the corresponding vertex from the Dynkin diagram. Thus when an extremal vertex is deleted the system $(\tilde{\mathcal{G}}^{(k)}, c_m)$ represents a single residual basic Lie algebra. Removing the affine vertex obviously gives $\mathcal{G}$. On the other hand deleting internal vertices splits the Dynkin diagram into two sections, corresponding to a pair of basic Lie algebras. For example, splitting $A_{2n}^{(2)}$ at the m^{th} vertex gives Lax operators in the B_{n-m} and C_m series. Furthermore the (pseudo)differential operator associated with the D_n series is derived using the embedding algebra $D_n^{(1)}$. In both situations a pair of (pseudo)differential operators is found whose product gives the Lax operator L of Eq. (7 1 3).

To determine the exact form of the vacuum solution ϕ recall that negative simple roots are lowering operators on system eigenstates. Further removal

of a root in the affine system produces the simple root system of a non affine Lie algebra. Thus, due to the linear combination among the roots of the affine system one root must be singled out to act as a conventional state raising operator. This role is given to the removed root. Thus, the vacuum eigenstate will be annihilated by a vacuum projection operator Δ^- defined by

$$\Lambda = \sum_{i=0}^{r} E_{i\neq m} + \lambda E_m \equiv \Delta^- + \lambda E_m \tag{7 1 7}$$

This requirement fixes the scalar vacuum solution ϕ by setting it equal to a linear combination of the components of $\vec{\psi}$ such that

$$\Delta^- \vec{\psi} = 0 \tag{7 1 8}$$

is satisfied. A direct relation between the scalar operator L and the matrix operator $\mathcal{L}$ will be given in the next section.

For the kernel equation (7 1 6) to produce a unique solution, we require that the number of independent degrees of freedom equal the rank of the embedding affine Lie algebra $\hat{\mathcal{G}}^{(k)}$ or equivalently the residual system $\{\mathcal{G}^{(k)}_{-cm}\}$. The extra degrees of freedom generate gauge invariance. To fix the gauge invariance, one must find a matrix operator $S(x)$ that enforces the gauge transformation

$$\mathcal{L}_0 = e^{ad\, S}\mathcal{L}. \tag{7 1 9}$$

where ad is the adjoint mapping. The gauge freedom present in Eq. (7 1 6) allows one freedom in determining the form of the coordinate dependent term $q_0(x)$, i.e.

$$\mathcal{L}_0 = I\frac{\partial}{\partial x} + \Lambda + q_0(x) \tag{7 1 10}$$

Drinfel'd and Sokolov find the sufficient condition that $S \in C^\infty(\mathbb{R}, \eta)$, where η is generated by the positive simple roots E_i, $i \neq m$.

Many authors including Drinfel d and Sokolov, work most frequently in the "canonical" gauge. However here we choose to work in their "diagonal" gauge which has the form

$$q^{diag}(x) = \begin{pmatrix} q_1 & 0 & & 0 & 0 \\ 0 & q_1 & & 0 & 0 \\ & & & & \\ 0 & 0 & & q_{N-1} & 0 \\ 0 & 0 & & 0 & q_N \end{pmatrix} \quad (7.1.11)$$

This gauge leads to the convenient form

$$q^{diag} = \sum_{i=0}^{r} v_i(q_1 q_2 \cdots q_N) H_i \quad (7.1.12)$$

which is in the canonical or principal gradation [82]. Here, H_i are the generators of the Cartan subalgebra and the functions v_i are linear combinations of the elements q_i. In this gauge, the gauge term q^{diag} associated with $(\mathcal{C}^{(k)}\ c_m)$ is the special case where the sum excludes $i = m$. The Lax operator L^{diag} generates the mKdV equations, and is related to L^{can} via the well known Miura transformations.

7.2 Diagrammatic Construction of Lax Operators

To exploit gauge invariance of the Lax operators, one should choose a $q(x)$ gauge most suited to ones needs. Here we are interested in developing a diagrammatic scheme for constructing L. In this regard, the diagonal gauge proves more useful than the other choices. In this section we will demonstrate how the diagonal gauge allows one to build Lax operators directly from cyclic weight diagrams of representations of affine Lie algebras.

To motivate the algorithm we first review the construction of L by solving the matrix system $\mathcal{L}\vec{\phi} = 0$. For the present discussion, it will be sufficient to

consider embedding algebras of the form $(\hat{\mathcal{G}}^{(k)}, c_0)$ where the affine vertex is deleted Thus the diagonal gauge simply reduces to the form

$$\psi^{diag}(x) = \sum_{i=1}^{r} v_i(x) H_i \tag{7.2.1}$$

where we have excluded H_0 from the sum

Consider again the canonical representation of the embedding affine lie algebra $(A_2^{(1)}, c_0)$ presented in Fig. 18 Matrix representations of the Cartan matrix can be read-off from the cyclic weight diagram. The matrix element $(H_i)_{jj}$ is extracted from the i^{th} element of the j^{th} weight vector, while the off diagonal elements are set to zero The matrix entry of the negative simple root $(E_i)_{jk}$ is assigned its normalisation factor if the k^{th} weight vector branches into the j^{th} weight vector as a result of subtracting the i^{th} row of the Cartan matrix in the process The other entries are by default zero Thus, the matrix representations of the simple roots are easily found to give

$$A = \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{7.2.2}$$

where the effect of the affine root indicated in Fig. 18 by the dashed arrow line, is assigned the value λ Plugging these values into the kernel equation (7.1.6) produces the system of equations

$$\begin{aligned} [\partial + v_1]\psi_1 &= -\lambda\psi_3 \\ [\partial - v_1 + v_2]\psi_2 &= -\psi_1, \\ [\partial - v_2]\psi_3 &= -\psi_2 \end{aligned} \tag{7.2.3}$$

Here on the right-hand side we have placed the terms due to the matrix A

The vacuum condition (7.1.8) determines the scalar function to be $\phi = \psi_3$ Thus, we must solve by starting with the last equation First, we multiply this equation through by $[\partial - v_1 + v_2]$ and then eliminate ψ_2 using the second

equation. Then, multiplying through by $[\partial + v_1]$ and using the top equation gives the scalar Lax eigenvalue equation

$$\begin{aligned} L^{[A_2^{(1)}\,\mathrm{cd}]}\psi_3 &= [\partial + v_1][\partial - v_1 + v_2][\partial - v_2]\psi_3 \\ &= -\lambda v_2 \end{aligned} \tag{7.2.4}$$

where the spectral parameter is given by $\mu = -\lambda$. Imposing the field redefinitions

$$\varphi_1 = v_1 \qquad \varphi_2 = v_2 - v_1 \tag{7.2.5}$$

gives the standard form

$$L^{[A_2^{(1)}\,\mathrm{cd}]} = [\partial + \varphi_1][\partial + \varphi_2][\partial - \varphi_1 - \varphi_2]. \tag{7.2.6}$$

This example exhibits a common feature relevant for our scheme below. When the vacuum condition (7.1.8) imposes the scalar eigenfunction to be given by a single component of the eigenfunction, say $\varphi = \psi_i$, then the resulting characteristic equation satisfies

$$L\varphi = \mu\varphi. \tag{7.2.7}$$

Consequently, the system reduction must start with the $i^{\underline{\mathrm{th}}}$ equation in the matrix system and proceed upward till the top equation is reached. If $i \leq N$ the process continues with the bottom equation and moves upward until the $i^{\underline{\mathrm{th}}}$ equation is reached again. We shall refer to this case as trivial since the corresponding cyclic affine weight diagram is linear containing no branch points. A second feature brought out in this example, is that the number of factors in the resultant Lax operator (7.2.6) is equal to the number of weights in the weight diagram. Unfortunately, this is valid only for trivial cases. Nevertheless this last observation is key to our scheme.

To highlight one more property of the general procedure we turn to a non trivial example. For this, we require a representation of an affine Lie algebra whose cyclic weight diagram has at least one branching point. Thus, consider the canonical representation of the affine algebra $(D_4^{(1)}, c_0)$. Fig. 21 presents the Dynkin diagram, Cartan matrix and corresponding cyclic weight diagram which has two branch points.

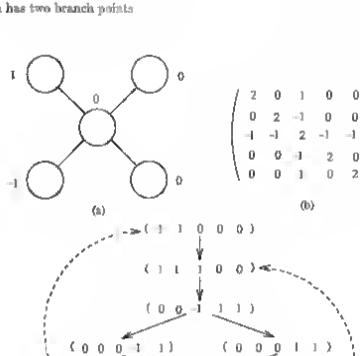


Figure 21. (a) Dynkin diagram, (b) Cartan matrix and (c) cyclic weight diagram of the (−11000) representation of $D_4^{(1)}$. The dashed arrows are the deleted root.

Reading off from the weight diagram gives

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{7.2.8}$$

The branch points have manifested themselves by placing more than one non zero entry in the second and sixth rows. Now further reading off the elements of the Cartan matrices gives the system of equations

$$\begin{aligned} [\partial + v_1]\psi_1 &= -\lambda\psi_7, \\ [\partial - v_1 + v_2]\psi_2 &= -\psi_1 - \lambda\psi_8, \\ [\partial - v_2 + v_3 + v_4]\psi_3 &= -\psi_2 \\ [\partial - v_3 + v_4]\psi_4 &= -\psi_3 \\ [\partial + v_3 - v_4]\psi_5 &= -\psi_3 \\ [\partial + v_3 - v_4]\psi_6 &= -\psi_4 - \psi_5 \\ [\partial + v_1 - v_3]\psi_7 &= -\psi_6, \\ [\partial - v_1]\psi_8 &= -\psi_7 \end{aligned} \tag{7.2.9}$$

The vacuum condition (7.1.8) produces two distinct solutions ψ_8 and the linear combination $\psi_4 - \psi_5$. Here we consider the first case. Proceeding as before, we eliminate ψ_1 and ψ_7 in the last two equations to get

$$[\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1]\psi_8 = -\psi_4 - \psi_5 \tag{7.2.10}$$

Now, we encounter a well-known technical problem not found in the trivial case. The components ψ_4 and ψ_5 can not both be simultaneously eliminated since the expressions $[\partial - v_2 + v_4]$ and $[\partial + v_3 - v_4]$ do not commute. This dilemma is directly linked to the fact the corresponding cyclic weight diagram

has a branch point connecting the fourth and fifth weights to a single weight located below them

To overcome this obstacle the pseudo-differential operator ∂_x^{-1} must be introduced Its operation on any function $f(x)$ is given by the expansion

$$\partial_x^{-1} f(x) = \sum_{n=0}^{\infty} (-1)^n f^{(n)}(x) \partial_x^{-n-1} \tag{7.2.11}$$

Utilizing the pseudo-differential operator we rewrite the fourth equation in (7.2.9) as

$$\psi_4 = -[\partial - v_3 + v_4]^{-1} \psi_5 \tag{7.2.12}$$

Thus the combined effect of the bottom five equations is

$$\begin{aligned} \psi_2 = \{[\partial - v_3 + v_4]^{-1} + [\partial + v_2 - v_4]^{-1}\}^{-1} [\partial + v_2 - v_3 - v_4] \\ \times [\partial + v_1 - v_3][\partial - v_1] \psi_5 \end{aligned} \tag{7.2.13}$$

A helpful identity we use repeatedly is

$$\{A^{-1} + B^{-1}\}^{-1} = \{A^{-1}[A + B]B^{-1}\}^{-1} = B[A + B]^{-1} A. \tag{7.2.14}$$

When applied to Eq (7.2.13) a cancellation occurs among the v_i's appearing in the curly brackets This simplifies the expression to

$$\psi_2 = \tfrac{1}{2}[\partial - v_3 + v_4]\partial^{-1}[\partial + v_3 - v_5][\partial + v_3 - v_4][\partial + v_1 - v_3][\partial - v_1]\psi_5 \tag{7.2.15}$$

Continuing, incorporating the next two equations in (7.2.9) requires a second application of the relation (7.2.14) Finally the Lax operator based on $(D_4^{(1)}, e_0)$ with vacuum ψ_5 is

$$\begin{aligned} L = \frac{1}{4}\partial^{-1}[\partial + v_1][\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4] \\ \times \partial^{-1}[\partial + v_3 - v_4][\partial + v_3 - v_2 - v_4][\partial + v_1 - v_3][\partial - v_1] \end{aligned} \tag{7.2.16}$$

Using the field redefinitions

$$\begin{aligned} q_1 &= v_1, & q_3 &= v_3 - v_1, \\ q_2 &= -v_2 + v_3 + v_4 & q_4 &= -v_3 + v_4, \end{aligned} \tag{7.2.17}$$

we get

$$L = \frac{1}{4}\partial^{-1}[\partial+q_1][\partial+q_2][\partial+q_3][\partial+q_4]\partial^{-1}[\partial-q_4][\partial-q_3][\partial-q_2][\partial-q_1] \quad (7.2.18)$$

which is proportional to the standard result

We have chosen these two examples because they introduce the techniques needed to generate Lax operators associated with even the most complicated algebraic systems. Furthermore they show how closely the structure of cyclic affine weight diagrams is linked with the construction of general Lax operators. As a result, we propose a set of four steps which allows one to construct Lax operators associated with cyclic representations of affine Lie algebras.

First, we propose that to every weight vector of an affine cyclic weight diagram one can associate an operator as follows

Step 1 $(a_0 a_1 a_2 \ldots) \rightarrow [\partial_x + a_1 v_1(x) + a_2 v_2(x) + \quad]$

The coefficient a_0 does not appear on the right hand side as it corresponds to the deleted vertex. Next we introduce a step which is designed to facilitate the construction of Lax operators when branch points exist in the corresponding weight diagram. Essentially, this step reduces more complicated non-trivial cases to a sum of manageable trivial cases by reducing the branched weight diagram to a sum a linear subdiagrams.

Step 2. Replace branching weight diagrams by the sum of linear subdiagrams each representing a vertical route beginning with, and ending on the vacuum weight(s)

For example, Fig. 22 presents the four linear subgraphs associated with the canonical representation of $D_4^{(1)}$

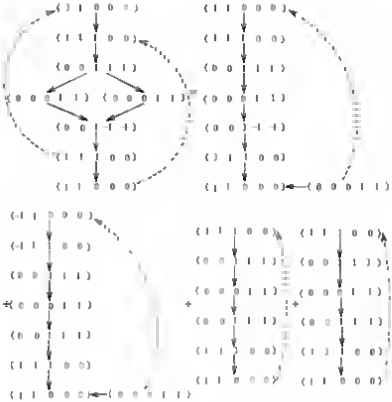


Figure 22 Subdiagrams of the cyclic weight diagram of $D_4^{(1)}$ The dashed arrows are the deleted root

In drawing cyclic weight diagrams it is important that the arrows generated by the deleted vertex are distinguished from the others Our convention is to use dashed lines Furthermore the direction of the arrows must also be noted The Lax operators associated with each subdiagram are then constructed as follows

Step 3– Circulate around the loop beginning with the vacuum solution, such that the flow is opposite most of the arrows

If a weight vector is approached by an arrow's

a) tail, append its weight factor to the operator's left side,

b) head, append the weight factor's inverse to the operator's left side

For weights at the tail end of both connecting arrows, do nothing

Multiply by the product of the corresponding normalization factors

The loop is to be circulated in a direction opposite most of the arrows so that the leading term of the Lax operator $L = \partial^n + \ldots$ has positive exponent, i.e., $n > 0$.

For trivial cases, this completes the computation of L. However, for non trivial cases with branching weight diagrams we can not naively build the final Lax operator from a sum of its constituent linear subgraphs. Instead, as we shall prove in the next section, they are added together analogously to how one computes total resistance of resistors in parallel.

Step 4. The Lax operator is given by the inverse of the sum of the inverses of the operators resulting from step three.

For example, in the non trivial case $\{D_1^{(1)}, c_2\}$ with $\phi = \psi_3$ as before, we build four operators corresponding to the linear subdiagrams in Fig. 22.

$$L_1 = [\partial - v_1]^{-1}[\partial + v_1][\partial - \iota_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4] \times [\partial + v_4 - v_3 - v_4][\partial + v_1 - \iota_2][\partial - v_1] \tag{7.2.19}$$

$$L_2 = [\partial - v_1]^{-1}[\partial + v_1][\partial - v_1 + v_3][\partial - v_3 + v_3 + v_4][\partial + v_3 - v_4] \times [\partial + v_3 - v_3 - \iota_4][\partial + \iota_1 - v_2][\partial - v_1], \tag{7.2.20}$$

$$L_3 = [\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4][\partial + v_3 - v_3 - v_3] \times [\partial + v_2 - v_3 - \iota_3]. \tag{7.2.21}$$

and

$$L_4 = [\partial - v_2 + v_3][\partial - v_2 + v_3 + v_4][\partial + v_3 - v_4][\partial + v_2 - v_3 - v_3] \times [\partial + v_2 - v_2][\partial - v_1] \tag{7.2.22}$$

Factoring out common terms we find

$$\begin{aligned} L^{-1} &= \sum_{i=1}^{4} (L_i)^{-1} \\ &= [\partial - v_1]^{-1}[\partial + v_1 - v_2]^{-1}[\partial + v_2 - v_3 - v_4]^{-1} \\ &\quad \times \{[\partial - v_3 + v_4]^{-1} + [\partial + v_3 - v_4]^{-1}\}[\partial - v_2 + v_3 + v_4]^{-1} \\ &\quad \times [\partial - v_1 + v_2]^{-1}[\partial + v_1]^{-1}\{1 + [\partial - v_1]^{-1}[\partial + v_1]\} \end{aligned} \tag{7 2 23}$$

By taking the reciprocal and simplifying we reproduce the previous result Eq (7 2 16)

To end this section we consider the alternative vacuum choice $\psi_3 - \psi_4$ It should be obvious that since we are dealing with cyclic weight diagrams, Lax operators associated with other vacuum states can be achieved by cyclically permuting factors in the primary Lax operator Thus this second vacuum choice immediately gives the Lax operator

$$L = \frac{1}{4}[\partial - g_3][\partial - g_2][\partial - g_1]\partial^{-1}[\partial + g_1][\partial + g_2][\partial + g_3]\partial^{-1}[\partial - g_4] \tag{7 2 24}$$

CHAPTER 8
PROOF OF DIAGRAMMATIC SCHEME

To prove the equivalence between the matrix system $\mathcal{L}\vec{\psi} = 0$ and the diagrammatic algorithm we begin by rewriting the former as

$$\mathcal{D}\vec{\psi}(x) = -\Lambda\vec{\psi}(x) \tag{8.1}$$

where to simplify notation we have defined

$$\mathcal{D} \equiv \mathcal{L} - \lambda = I\frac{\partial}{\partial x} + q(x) \tag{8.2}$$

The structure of the associated cyclic weight diagram is encoded entirely in the matrix Λ. Specifically, recall that the general matrix element Λ_{ij} is proportional to λ if the difference between the i^{th} weight and the connecting j^{th} weight equals the eliminated root of the embedding affine simple root system. All other connecting weights Λ_{ij} are proportional to 1. Otherwise the matrix element is assigned the value 0. In all cases the proportionality constant is the normalization factor of the connecting root.

We construct the proof in stages. For the first stage we consider the trivial case i.e. a single entry in each row and column of Λ. Removing a root produces only a single vacuum state. This stage corresponds to weight diagrams with no branch points and only one arrow associated with the eliminated root. Clearly, we can rearrange the matrix equations as $\mathcal{L}\vec{\psi} = 0$ such that the vacuum state equation appears last. Furthermore, it can be arranged such that Λ is lower triangular with ones located along a diagonal once removed from the main

diagonal except for the eliminated root whose coefficient λ appears in the upper right hand corner, i.e. A is a circulant matrix. Thus, the Lax eigenvalue equation becomes

$$L\psi_N(x) = \mu\psi_N(x) \tag{5.3}$$

Since $\mathcal{D}$ is diagonal, the k^{th} equation in (5.1) can be written

$$\mathcal{D}_k\psi_k = -\sum_{i=1}^{N} \Lambda_{ki}\psi_i, \qquad k = 2, 3, \ldots, N \tag{5.4}$$

Clearly, since A is a circulant matrix as specified above, the inequality $i < k$ holds for $k \neq 1$. Next, by repeatedly replacing the function ψ_i, appearing on the right hand side with the i^{th} matrix equation we eventually reach the expression $\Lambda_{k1}\psi_1$, i.e.

$$\begin{aligned}
\mathcal{D}_N\psi_N &= -\sum_{i=1}^{\Lambda} \lambda_i \psi_i \\
&= \sum_{i=1}^{N}\sum_{j=1}^{\Lambda} \Lambda_{Ni}\mathcal{D}^{-1}\Lambda_{ij}\psi_j \\
&= (-)^{\gamma}\sum_{i=1}^{\Lambda} \sum_{k=1}^{\Lambda} \Lambda_{Ni}\mathcal{D}_i^{-1}\Lambda_{ij} \cdots \mathcal{D}_k^{-1}\Lambda_{k1}\psi_1
\end{aligned} \tag{5.5}$$

where γ is the height of the cyclic weight diagram. Due to the successive applications of the state lowering operators Λ_{ij} with $i > j$, this equation is interpreted as taking the highest state ψ_1 and lowering it to the vacuum state ψ_N.

Replacing ψ_1 through

$$\mathcal{D}_1\psi_1 = -\lambda_0\psi_\Lambda \tag{5.6}$$

where we have used $A_{1N} = \mathcal{N}_0\lambda$ which excites the level of the state since $1 < N$, gives

$$\mathcal{D}_N\psi_N = -(-)^2\mathcal{N}_0\lambda\sum_{i=1}^{N}\sum_{k=1}^{N}A_{Ni}\mathcal{D}_i^{-1}A_{ij}\mathcal{D}_j^{-1}A_{k1}\mathcal{D}_1^{-1}\psi_N \tag{8.7}$$

where $\mathcal{N}_0$ is the normalization factor of the affine root. Finally moving terms to the left hand side we recover (8.3) where

$$L = \{\mathcal{N}_0\sum_{i=1}^{N}\sum_{k=1}^{N}\mathcal{D}_N^{-1}A_{Ni}\mathcal{D}_i^{-1}\quad A_{k1}\mathcal{D}_1^{-1}\}^{-1} \tag{8.8}$$

and the spectral parameter is given by

$$\mu = -(-)^N\lambda \tag{8.9}$$

Since each row and column of A contain only one entry the sum will generate a single term, i.e

$$L^{-1} = \{\mathcal{N}\mathcal{D}_N^{-1}\mathcal{D}_{N-1}^{-1}\quad \mathcal{D}_1^{-1}\}^{-1} \tag{8.10}$$

where $\mathcal{N}$ is the product of the normalization factors. Now each $\mathcal{D}$ is a weight factor as defined in step 1. Thus there is a direct mapping between the order of the weight factors and their location in the corresponding weight diagram.

Now suppose we permit multiple row entries in A i.e., branch points in the weight diagram. First consider the case where such multiple entries occur above the N^{th} row. As before, there is a single vacuum state, and the constant λ is located in the upper right hand corner of A. Therefore the constraint $i < k$, for $k \neq 1$ remains in effect for (8.4). Hence the derivation leading to (8.8) follows through unchanged. Now each new entry in A causes an additional final term in (8.8). Clearly per step 4 of the diagrammatic algorithm, the final Lax operator is obtained by taking the reciprocal of the sum of terms generated by (8.8).

Next, suppose the multiple row entries in Λ, due to the branch point, occur in the N^{th} row. The vacuum condition (7.1.8) shows that this is equivalent to a degenerate vacuum state with, say, degeneracy d. Subsequently, this row will be associated with the eliminated root, and the d integers will be assigned the value λ. In fact, λ appears only in this row. Clearly, in the weight diagram the d weights share the same level.

Let us first discuss the case where the coefficients λ occur in the first row. Thus, Eq. (8.4) remains valid, keeping intact the constraint $s < k$, for $k \neq 1$. Furthermore, the scalar eigenfunction $\phi(\lambda)$ is now a linear combination of the components $\psi_N, \psi_{N-1}, \ldots, \psi_{N-d+1}$, and the equation for ψ_1 becomes

$$\mathcal{D}_1 \psi_1 = -\sum_{s=N-d+1}^{N} \Lambda_{1s} \psi_s \tag{8.11}$$

Consequently, eq. (8.7) is modified to

$$\psi_s = -(-)^\beta \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=N-d+1}^{\Lambda} \mathcal{D}_s^{-1} \Lambda_{sk} \mathcal{D}_i^{-1} \times \mathcal{D}_i^{-1} \Lambda_{1i} \mathcal{D}_1^{-1} \Lambda_{1j} \psi_j \tag{8.12}$$

where β is the number of field replacements performed. The characteristic equation is obtained by multiplying both sides by Λ_{1s} and then summing over $s = 1, \ldots, \Lambda$:

$$\sum_{s=N-d+1}^{N} \Lambda_{1s} \psi_s = -(-)^\beta \sum_{s=1}^{\Lambda} \sum_{k=1}^{\Lambda} \sum_{j=N-d+1}^{N} \Lambda_{1s} \mathcal{D}_s^{-1} \Lambda_{sk} \times \mathcal{D}_k^{-1} \Lambda_{k1} \mathcal{D}_1^{-1} \Lambda_{1j} \psi_j \tag{8.13}$$

Note, the sum over s on the right-hand side has been extended to the entire range for convenience.

Each term in Λ_{1s} contains the factor λ, which can then be factored out. As a result, the scalar eigenfunction is found to be

$$\phi = \sum_{s=\Lambda-d+1}^{\Lambda} \Lambda_{1s} \psi_s \tag{8.14}$$

and the Lax operator

$$L = \lambda\{\sum_{i=1}^{N} \sum_{k=1}^{\Lambda} \Lambda_{ik}\mathcal{D}_i^{-1}\Lambda_{ik} \quad \Lambda_{kl}\mathcal{D}_1^{-1}\}^{-1} \tag{8 15}$$

Clearly, this has the same interpretation as the non degenerate branching case

For the last stage of the proof we relax the condition that multiple occurrences of λ must all be in the first row of Λ In the weight diagram this means not all the arrows associated with the eliminated root point to the bottom level Recall from the discussion surrounding eq (7 1 7), the eliminated root with coefficient λ acts as a state raising operator Thus, every occurrence of λ will appear in the upper triangular portion of Λ, and the unit coefficients of the state lowering roots are in the lower triangular portion

For λ in the $k^{\underline{\text{th}}}$ row of Λ $k \neq 1$ Eq (8 4) is modified to

$$\mathcal{D}_j\psi_j = -\Lambda_{jk}\lambda\psi_k - \sum_{n\neq k}^{\Lambda} \Delta^-_{jn}\psi_n \tag{8 16}$$

where, since λ corresponds to the state raising operator, $k > j$ Consider the case where j is the largest such index to satisfy this equation. Then, allowing degenerate vacuum states we have

$$\begin{aligned}\psi_s &= -(-)^\beta \sum_{i=1}^{N} \sum_{j=1}^{\Lambda} \mathcal{D}_i^{-1}\lambda \quad \mathcal{D}^{-1} \quad \Lambda_{ij}\psi_j \\ &= (-)^\beta \sum_{i=1}^{N} \sum_{j=1}^{\Lambda} \mathcal{D}_i^{-1}\Lambda_{ik}\mathcal{D}_1^{-1} \quad \Lambda_{ij}\mathcal{D}_1^{-1}[\mathcal{N}_0\lambda\psi_k + \sum_{n\neq k}^{N} \Delta^-_{jn}\psi_n],\end{aligned} \tag{8 17}$$

where, again, β is the number of field replacements performed The effect of the factor in front of ψ_k is to first due to λ, raise this state to ψ_j and then to lower it till the vacuum state ψ_s is reached Now, since the corresponding weight diagram is cyclic there must exist some factor that will circulate ψ_s back to ψ_k

First, as was the case with $\{D_i^{(i)} \; c_0\}$ consider the situation where ψ_k is an intermediate state in (8.17) i.e.

$$\psi_i = (-)^n \sum_{i=1}^{N} \sum_{k=1}^{N} \mathcal{D}_i^{-1} \Lambda_{ik} \mathcal{D}_j^{-1} \quad \Lambda_{jk} \psi_k \tag{8.18}$$

This gives

$$\psi_i = (-)^n \{ \sum_{i=1}^{N} \sum_{k=1}^{N} \mathcal{D}_i^{-1} \Lambda_{ik} \mathcal{D}^{-1} \quad \Lambda_{jk} \}^{-1} \psi_k \tag{8.19}$$

Thus, per step 3 of the diagrammatic algorithm the factor $\mathcal{D}_k$ associated with the weight vector at the tail end of both connecting arrows does not appear. Further, proceeding from higher weights to lower weights in the weight diagram contributes factors of $\mathcal{D}_i^{-1}$ in the operator defined in step 2 for the linear subdiagram.

Finally, if ψ_k does not appear as an intermediate state of the vacuum state ψ_i in (8.17) then it must occur as an intermediate state for one of the other vacuum states. Again, since the weight diagram is cyclic, there is some closed path going from ψ_i to each of the other vacua. However, to write down a final expression is too unwieldy. Nevertheless, it should be clear that the general rules of the diagrammatic algorithm are complete and provide an accurate mapping between weight diagrams and the scalar Lax equation.

CHAPTER 9
SUPERSYMMETRIC LAX OPERATORS

In this section we generalize the diagrammatic scheme to supersymmetric affine Lie algebras. The classification of all possible supersymmetric extensions of the basic Lie algebras has been given by Kac [83]. In addition to the bosonic simple roots of the basic Lie algebra, the simple root system of the supersymmetric algebras contains two distinct kinds of fermionic roots. The Dynkin symbol of the first type is sometimes given by a shaded vertex representing a non zero norm. The second fermionic root type has zero norm whose Dynkin symbol is given correspondingly by a crossed out vertex. As always, the bosonic root is denoted by a white vertex.

A new feature occurring in the supersymmetric Lie algebras is that they may have several non equivalent simple root systems corresponding to different Dynkin diagrams and Cartan matrices. In other words, the different root systems can not be transformed into each other through standard Weyl rotations. Instead, they are obtained by performing the "Weyl" transformation with respect to the nilpotent fermionic root. For more details, see Frappat *et al.* [84] which also presents a large collection of Dynkin diagrams associated with all of the classical contragradient supersymmetric cases, those of the affine and twisted affine supersymmetric algebras.

Non-equivalent simple root systems which represent the same supersymmetric Lie algebra differ in the distribution of bosonic and fermionic roots. However, here we are interested in considering a natural extension of the

Drinfel'd-Sokolov procedure to the supersymmetric case. This restricts our choice of the simple root system for building the supersymmetric Lax operators [85]. Recall in the bosonic case the mKdV Lax operator constructed with the gradation choice of Drinfel'd and Sokolov generates Toda lattice models [23,26,87]. For supersymmetric algebras it has been shown [88] that Toda lattices are possible only for simple root systems composed purely of fermionic roots. Supersymmetric Lie algebras with purely fermionic root systems have been given by Leites *et al.* [89]

$$\begin{cases} SL(n+1 \mid n) \\ OSp(m \mid 2n) \\ D(2 \mid 1 \ \alpha) \end{cases} \quad \text{for } m = 2n,\ 2n+2,\ 2n \pm 1 \tag{9.1}$$

Also, the infinite-dimensional affine supersymmetric Lie algebras with purely fermionic simple root systems are

$$\begin{cases} SL(n \mid n)^{(1)} \\ OSp(2n+2 \mid 2n)^{(1)}, \\ D(2 \mid 1 \ \alpha)^{(1)} \end{cases} \tag{9.2}$$

while the infinite-dimensional twisted affine cases are

$$\begin{cases} SQ(2n+1)^{(4)} \\ SL(n \mid n)^{(2)} \\ OSp(2n \mid 2n)^{(2)} \end{cases} \tag{9.3}$$

The supersymmetric extension of the KdV equations was first discussed in Manin and Radul 90. They suggested replacing the bosonic derivative ∂_x by its supersymmetric analog D:

$$\partial_x \to D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x} \tag{9.4}$$

Note that $D^2 = \frac{\partial}{\partial x}$. The system of matrix equations of Drinfel'd and Sokolov can then be generalized to [85]

$$\mathcal{L}\vec{\psi}(x,\theta) \equiv [D + Q(x,\theta) + \Lambda]\vec{\psi}(x,\theta) = 0 \tag{9.5}$$

where A is generated by the purely negative fermionic roots, and $Q(x,\theta)$ is a Grassmann odd fermionic superfield which can be expanded as

$$Q(x,\theta) = \sum_{i=1}^{r} \overline{H}_i \Psi_i(x,\theta) \tag{9.6}$$

where now H_i are elements of the Cartan Kac subalgebra. The vacuum condition is as before,

$$\Delta^{-}\vec{\psi}(x,\theta) = 0 \tag{9.7}$$

Since the second type of fermionic root is nilpotent, they deserve special treatment when constructing their weight diagrams. To illustrate how this comes about, consider the fundamental representation of the supersymmetric algebra $OSp(2\,|\,2)$. The Dynkin diagram of the purely fermionic root system is given by

where both fermionic roots are depicted as having zero norm, and where we have indicated the choice (11) for a highest weight vector. The Cartan matrix is then easily found to be

$$A = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \tag{9.8}$$

To construct the highest weight diagram we proceed as before. Since a positive one appears both in the first and second places we have two permutations of subtraction to perform. In particular, we can start by subtracting the first row of the Cartan matrix giving (13), and then subtracting the second row resulting in (33). However, unlike the bosonic case, we may not subtract the first row of the Cartan another time from the weight (13). This is because here the fermionic weight vectors are nilpotent and subtracting any Cartan row twice gives a decoupled state. Similarly, we can start by subtracting the

second Cartan row once(and only once) and then the first row giving (33) Thus, we find the weight diagram with height two

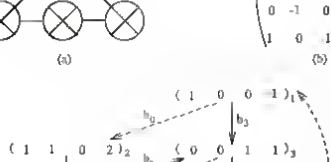
$$\begin{array}{ccc} & (11)_1 & \\ \swarrow & & \searrow \\ (13)_2 & & (31)_3 \\ \searrow & & \swarrow \\ & (33)_4 & \end{array} \qquad (9\,9)$$

The decoupling which occurs when constructing a cyclic weight diagram for an affine supersymmetric algebra is almost as straightforward For example in Fig 23

(a)

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

(b)



(c)

Figure 23 (a) Dynkin diagram (b) Cartan matrix, and (c) complete weight diagram of the (1001) representation of $SL(2\,|\,2)^{(1)}$ The dashed arrows are the deleted root The box encloses the cyclic weight diagram excluding the decoupled weights outside

we display the partially decoupled weight diagram of $(SL(2\,[\,2)^{(1)}, c_0)$ where states were decoupled as we went from top to bottom. There are several ways to decouple the remaining weights since the lowering operators b_1, b_2 and b_3 still appear more than once. The only way for a cyclic weight diagram to emerge is by decoupling the weights outside the box. To see that this is also consistent, note that all paths leading from weight 5 to weights 2 or 4 require two applications of b_0.

To construct a super-Lax operator, let us take the vacuum solution $\phi = \Psi_4$. We easily find the super-Lax operator to be

$$L = [D + \Psi_3][D + \Psi_2 + \Psi_3][D + \Psi_1 + \Psi_2 + \Psi_3][D + \Psi_4] \qquad (9.10)$$

As in the non-supersymmetric case, Lax operators corresponding to the other three vacua are obtained through cyclic permutations of the above operator.

We have shown how one can read off from cyclic weight diagrams associated with representations of affine Lie algebras Lax operators in the diagonal gauge. This method is most useful when tables of matrix representations are not at hand and must be generated by weight diagrams anyway. Furthermore, this procedure can easily be implemented on computer by virtue of the fact that computer generated algorithms currently exist for building highest weight diagrams [91]. With minor modifications, these programs can be adapted for cyclic weight diagrams.

It remains to be seen whether higher representations lead to any new physics. If so then a program of categorizing these results might be pursued to identify redundant solutions. This might be easier to answer for supersymmetric algebras since nilpotency projects out decoupled weight vectors. What is clear though, at least for the non-supersymmetric cases, is that these higher

representations lead to integrable systems. Recall, to prove the integrability of KdV systems Drinfel'd and Sokolov found the necessary infinite set of conserved currents to be given by the coefficients of the Laurent expansion of $\mathcal{L}$ in the affine parameter λ. Our conclusion follows from the fact that every representation of a basic Lie algebra has an affine extension and that defining properties of affine Lie algebras are representation independent.

It would be interesting to see if our procedure could be modified to directly generate Lax operators in other gauges. Furthermore in light of recent work [82] on generalizations of the Drinfel'd and Sokolov scheme, one may also consider different gradations of the affine Lie algebra from which to obtain the matrix A and the form of $q(\lambda)$.

CHAPTER 10
CONCLUSION

We have shown how using the renormalization group to two-loop order and without making restrictive assumptions, since we ran the parameters of our theories numerically allowed us to probe a very promising extension of the standard model namely the supersymmetric one where patterns, *hitherto* unverifiable can be made more plausible and quantitative as the low energy data of the standard model keep improving.

We saw how the renormalization group gave us some good predictions for the range of masses for the top quark and Higgs boson. These limits may very well be tested experimentally very soon with the existing particle accelerators or those in construction. Discovery of one of these particles would help in better selecting candidate theories for extending the standard model.

Also the renormalization group gave us a way to approach the fermions mass problem by enabling us to put to the test various Ansatze for the structure of the Yukawa sector. In particular we observed that supersymmetry helps significantly making more credible some commonly studied "textures" among the Yukawa couplings.

En résumé we have witnessed how the renormalization group allowed us to believe that supersymmetry could be a rather elegant way to extend the standard model.

APPENDIX A
THE STANDARD MODEL β FUNCTIONS

In this appendix we compile the renormalization group β functions of the standard model [17]. These have appeared in one form or another in various sources. We have endeavored to confirm their validity through a comparative analysis of the literature. Our main source is Ref. 18. Following their conventions

$$\mathcal{L} = \overline{Q}_L \tilde{\Phi} \mathbf{Y}_u {}^\dagger u_R + \overline{Q}_L \Phi \mathbf{Y}_d {}^\dagger d_R + \overline{\ell}_L \Phi \mathbf{Y}_l {}^\dagger e_R + h.c. - \frac{1}{2}\lambda(\Phi^\dagger \Phi)^2 \quad , \tag{A.1}$$

where flavor indices have been suppressed and where Q_L and ℓ_L are the quark and lepton SU(2) doublets, respectively

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{A.2}$$

Φ and $\tilde{\Phi}$ are the Higgs scalar doublet and its SU(2) conjugate

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \tilde{\Phi} = i\tau_2 \Phi \tag{A.3}$$

u_R, d_R, and e_R are the quark and lepton SU(2) singlets and $\mathbf{Y}_{u\,d\,e}$ are the matrices of the up-type, down type and lepton type Yukawa couplings.

A.1 Gauge Couplings

The β functions for the gauge couplings are

$$\begin{aligned} \frac{dg_l}{dt} &= -b_l \frac{g_l^3}{16\pi^2} - \sum_k b_{kl} \frac{g_k^2 g_l^3}{(16\pi^2)^2} \\ &\quad - \frac{g_l^3}{(16\pi^2)^2} \mathrm{Tr}\{C_{lu} \mathbf{Y}_u {}^\dagger \mathbf{Y}_u + C_{ld} \mathbf{Y}_d {}^\dagger \mathbf{Y}_d + C_{le} \mathbf{Y}_e {}^\dagger \mathbf{Y}_e\} \ , \end{aligned} \tag{A.1.1}$$

where $t = \ln\mu$ and $i = 1\ \ 2\ \ 3$ corresponding to the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ of the Standard Model. The various coefficients are defined to be

$$\begin{aligned} b_1 &= -\frac{4}{3}n_g - \frac{1}{10} \\ b_2 &= \frac{22}{3} - \frac{4}{3}n_g - \frac{1}{6}\,, \\ b_3 &= 11 - \frac{4}{3}n_g\,, \end{aligned} \tag{A.1.2}$$

$$(b_{kl}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{136}{3} & 0 \\ 0 & 0 & 102 \end{pmatrix} - n_g \begin{pmatrix} \frac{19}{15} & \frac{1}{5} & \frac{11}{30} \\ \frac{3}{5} & \frac{49}{3} & \frac{3}{2} \\ \frac{44}{15} & 4 & \frac{76}{3} \end{pmatrix} - \begin{pmatrix} \frac{9}{10} & \frac{3}{10} & 0 \\ \frac{9}{10} & \frac{13}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.1.3}$$

and

$$(C_{lf}) = \begin{pmatrix} \frac{17}{10} & \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & 2 & 0 \end{pmatrix} \quad \text{with } f = u\,,\ d\,,\ e\,, \tag{A.1.4}$$

with $n_g = \frac{1}{2}n_{fl}$

A.2 Yukawas

In the Yukawa sector the β functions are

$$\frac{d\mathbf{Y}_{u,d,e}}{dt} = \mathbf{Y}_{u,d,e}\left(\frac{1}{16\pi^2}\beta^{(1)}_{u,d,e} + \frac{1}{(16\pi^2)^2}\beta^{(2)}_{u,d,e}\right), \tag{A.2.1}$$

where the one-loop contributions are given by

$$\begin{aligned} \beta^{(1)}_u &= \frac{3}{2}(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u - \mathbf{Y}_d{}^\dagger\mathbf{Y}_d) + 1_2(S) - (\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2) \\ \beta^{(1)}_d &= \frac{3}{2}(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d - \mathbf{Y}_u{}^\dagger\mathbf{Y}_u) + 1_2(S) - (\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)\,, \\ \beta^{(1)}_e &= \frac{3}{2}\mathbf{Y}_e{}^\dagger\mathbf{Y}_e + 1_2(S) - \frac{9}{4}(g_1^2 + g_2^2) \end{aligned} \tag{A.2.2}$$

with

$$Y_2(S) = \mathrm{Tr}\{3\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + 3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\}\,, \tag{A.2.3}$$

and the two-loop contributions are given by

$$\begin{aligned}
\beta_u^{(2)} = & \frac{3}{2}(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 - \mathbf{Y}_u{}^\dagger\mathbf{Y}_u\mathbf{Y}_d{}^\dagger\mathbf{Y}_d - \frac{1}{4}\mathbf{Y}_u{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + \frac{11}{4}(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 \\
& + Y_2(S)(\frac{5}{4}\mathbf{Y}_d{}^\dagger\mathbf{Y}_d - \frac{9}{4}\mathbf{Y}_u{}^\dagger\mathbf{Y}_u) - \chi_4(S) + \frac{3}{2}\lambda^2 - 2\lambda(3\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + \mathbf{Y}_d{}^\dagger\mathbf{Y}_d) \\
& + (\frac{223}{80}g_1^2 + \frac{135}{16}g_2^2 + 16g_3^2)\mathbf{Y}_u{}^\dagger\mathbf{Y}_u - (\frac{43}{80}g_1^2 - \frac{9}{16}g_2^2 + 16g_3^2)\mathbf{Y}_d{}^\dagger\mathbf{Y}_d \\
& + \frac{5}{2}\chi_4(S) + (\frac{9}{200} + \frac{20}{45}n_g)g_1^4 - \frac{9}{20}g_1^2g_2^2 + \frac{19}{15}g_1^2g_3^2 - (\frac{35}{4} - n_g)g_2^4 + 9g_2^2g_3^2 \\
& - (\frac{404}{3} - \frac{80}{3}n_g)g_3^4
\end{aligned}$$

$$\begin{aligned}
\beta_d^{(2)} = & \frac{3}{2}(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 - \mathbf{Y}_d{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u - \frac{1}{4}\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \frac{11}{4}(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 \\
& + Y_2(S)(\frac{5}{4}\mathbf{Y}_u{}^\dagger\mathbf{Y}_u - \frac{9}{4}\mathbf{Y}_d{}^\dagger\mathbf{Y}_d) - \chi_4(S) + \frac{3}{2}\lambda^2 - 2\lambda(3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_u{}^\dagger\mathbf{Y}_u) \\
& + (\frac{187}{80}g_1^2 + \frac{135}{16}g_2^2 + 16g_3^2)\mathbf{Y}_l{}^\dagger\mathbf{Y}_l - (\frac{79}{80}g_1^2 - \frac{9}{16}g_2^2 + 16g_3^2)\mathbf{Y}_u{}^\dagger\mathbf{Y}_u \\
& + \frac{5}{2}\chi_4(S) - (\frac{29}{200} + \frac{1}{45}n_g)g_1^4 - \frac{27}{20}g_1^2g_2^2 + \frac{31}{15}g_1^2g_3^2 - (\frac{35}{4} - n_g)g_2^4 \\
& + 9g_2^2g_3^2 - (\frac{404}{3} - \frac{80}{9}n_g)g_3^4
\end{aligned}$$

$$\begin{aligned}
\beta_e^{(2)} = & \frac{3}{2}(\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2 - \frac{9}{4}Y_2(S)\mathbf{Y}_l{}^\dagger\mathbf{Y}_e - \chi_4(S) + \frac{3}{2}\lambda^2 - 6\lambda\mathbf{Y}_e{}^\dagger\mathbf{Y}_e \\
& + (\frac{387}{80}g_1^2 + \frac{135}{16}g_2^2)\mathbf{Y}_l{}^\dagger\mathbf{Y}_e + \frac{5}{2}\chi_4(S) + (\frac{51}{200} + \frac{11}{5}n_g)g_1^4 + \frac{27}{20}g_1^2g_2^2 \\
& - (\frac{35}{4} - n_g)g_2^4 .
\end{aligned} \tag{A.2.4}$$

with

$$\begin{aligned}
\chi_4(S) = & (\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_l{}^\dagger\mathbf{Y}_l\} \\
& + (\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} + \frac{3}{4}(g_1^2 + g_2^2)\,\mathrm{Tr}\{\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} ,
\end{aligned} \tag{A.2.5}$$

and

$$\chi_4(S) = \frac{9}{4}\mathrm{Tr}\{3(\mathbf{Y}_u^\dagger\mathbf{Y}_u)^2 + 3(\mathbf{Y}_d^\dagger\mathbf{Y}_d)^2 + (\mathbf{Y}_e^\dagger\mathbf{Y}_e)^2 - \frac{2}{3}\mathbf{Y}_u^\dagger\mathbf{Y}_u\mathbf{Y}_d^\dagger\mathbf{Y}_d\} \tag{A 2 6}$$

A.3 Higgs' Vacuum Expectation Value and Self-Quartic Coupling

In the Higgs sector we present β functions for the quartic coupling and the vacuum expectation of the scalar field. The β function for the vacuum expectation value of the scalar field is

$$\frac{dh_0}{dt} = \frac{1}{16\pi^2}\gamma^{(1)} + \frac{1}{(16\pi^2)^2}\gamma^{(2)} \tag{A 3 1}$$

where the one-loop contribution is given by

$$\gamma^{(1)} = \frac{9}{4}(\frac{1}{5}g_1^2 + g_2^2) - Y_2(S)\,, \tag{A 3 2}$$

and the two-loop contribution is given by

$$\gamma^{(2)} = -\frac{3}{2}\lambda^2 - \frac{5}{2}Y_2(S) + Y_4(S) - (\frac{93}{800} + \frac{1}{2}\alpha_{ij})g_1^2 + (\frac{311}{32} - \frac{5}{2}\alpha_{ij})g_2^2 - \frac{27}{80}g_1^2g_2^2 \tag{A 3 3}$$

These expressions were arrived at using the general formulas provided in Ref. 18 for the anomalous dimension of the scalar field choosing the Landau gauge

Here we correct a discrepancy in the one-loop contribution to the quartic coupling of Ref. 18

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\beta_\lambda^{(1)} + \frac{1}{(16\pi^2)^2}\beta_\lambda^{(2)} \tag{A 3 4}$$

where the one loop contribution is given by

$$\beta_\lambda^{(1)} = 12\lambda^2 - (\frac{9}{5}g_1^2 + 9g_2^2)\lambda + \frac{9}{4}(\frac{3}{25}g_1^4 + \frac{2}{5}g_1^2g_2^2 + g_2^4) + 4Y_2(S)\lambda - 4H(S) \tag{A 3 5}$$

with

$$H(S) = \mathrm{Tr}\{3(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 + 3(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 + (\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2\}\ , \tag{A 3 6}$$

and the two loop contribution is given by

$$\begin{aligned}
\beta_\lambda^{(2)} = &-78\lambda^3 + 18\lambda(\frac{3}{5}g_1^2 + 3g_2^2)\lambda^2 - [\,(\frac{313}{8} - 10n_g)g_2^4 - \frac{117}{20}g_1^2g_2^2 \\
&+ \frac{9}{25}(\frac{220}{4} + \frac{50}{9}n_g)g_1^4\,]\lambda + (\frac{497}{8} - 8n_g)g_2^6 - \frac{3}{5}(\frac{97}{24} + \frac{8}{3}n_g)g_1^2g_2^4 \\
&- \frac{9}{25}(\frac{239}{24} + \frac{40}{9}n_g)g_1^4g_2^2 - \frac{27}{125}(\frac{59}{24} + \frac{40}{9}n_g)g_1^6 \\
&- 64g_3^2\mathrm{Tr}\{(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 + (\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2\} \\
&- \frac{8}{5}g_1^2\,\mathrm{Tr}\{2(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 - (\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 + 3(\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2\} \\
&- \frac{3}{2}g_2^2\tilde{Y}_4(S) + 10\lambda[\,(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} \\
&+ (\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} + \frac{3}{4}(g_1^2 + g_2^2)\,\mathrm{Tr}\{\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\}\,] \\
&+ \frac{3}{5}g_1^2[\,(-\frac{57}{10}g_1^2 + 21g_2^2)\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} + (\frac{3}{2}g_1^2 + 9g_2^2)\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} \\
&+ (-\frac{15}{2}g_1^2 + 11g_2^2)\,\mathrm{Tr}\{\mathbf{Y}_l{}^\dagger\mathbf{Y}_l\}\,] - 24\lambda^2\tilde{Y}_2(S) - \lambda\tilde{H}(S) \\
&+ 6\lambda\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\,\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} + 20\,\mathrm{Tr}\{3(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^3\} + 3(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^3 \\
&+ (\mathbf{Y}_e{}^\dagger\mathbf{Y}\,)^3\} - 12\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u(\mathbf{Y}_l{}^\dagger\mathbf{Y}_u + \mathbf{Y}_d{}^\dagger\mathbf{Y}_d)\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\}
\end{aligned} \tag{A.3 7}$$

A 4 Gauge Couplings and Masses in $SU(3)_C \times U(1)_{em}$

In the low energy regime the effective theory is $SU(3)_C \times U(1)_{em}$ We employ the general formula of Ref 92 to arrive at the β functions for the

respective gauge couplings

$$\frac{dg_3}{dt} = [\frac{2}{3}(n_u + n_d) - 11]\frac{g_3^3}{(4\pi)^2} + [\frac{38}{3}(n_u + n_d) - 102]\frac{g_3^5}{(4\pi)^4} + [\frac{8}{9}n_u + \frac{2}{9}n_d]\frac{g_3^3 e^2}{(4\pi)^4} + [\frac{5033}{18}(n_u + n_d) - \frac{325}{54}(n_u + n_d)^2 - \frac{2857}{2}]\frac{g_3^7}{(4\pi)^6}, \quad (A\,4\,1)$$

and

$$\frac{de}{dt} = [\frac{16}{9}n_u + \frac{4}{9}n_d + \frac{4}{3}n_l]\frac{e^3}{(4\pi)^2} + [\frac{64}{27}n_u + \frac{4}{27}n_d + 4n_l]\frac{e^5}{(4\pi)^4} + [\frac{64}{9}n_u + \frac{16}{9}n_d]\frac{e^3 g_3^2}{(4\pi)^4}, \quad (A\,4\,2)$$

where n_u, n_d and n_l are the number of up-type quarks, down type quarks and leptons, respectively. In Eq. (A 4 1) we have also included the three-loop pure QCD contribution to the β function of g_3 [93].

For the evolution of the fermion masses we used Ref. 94. It is known that there is an error in their printed formula [95]. Using the corrected expression, we compute the following mass anomalous dimension. The fermion masses in the low energy theory then evolve as follows:

$$\frac{dm}{dt} = \gamma_{(l,q)} m, \quad (A\,4\,3)$$

where the l and q refer to a particular lepton or quark, and where

$$\gamma_{(l,q)} = \gamma^1_{(l,q)}\frac{e^2}{(4\pi)^2} + \gamma^3_{(l,q)}\frac{g_3^2}{(4\pi)^2} + [\gamma^{11}_{(l,q)}e^4 + \gamma^{31}_{(l,q)}g_3^4 + 2\gamma^{13}_{(l,q)}e^2 g_3^2]\frac{1}{(4\pi)^4} + \gamma^{33}_{[q]}\frac{g_3^6}{(4\pi)^6} \quad (A\,4\,4)$$

The superscripts 1 and 3 refer to the $U(1)_{em}$ and $SU(3)_C$ contributions, respectively. Explicitly the above coefficients are given by

$$
\begin{aligned}
&\gamma^1_{(l,q)} = -6Q^2_{(l,q)} \\
&\gamma^3_{(l)} = 0 \\
&\gamma^3_{(q)} = -8 \\
&\gamma^{13}_{(l)} = \gamma^{33}_{(l)} = 0 \\
&\gamma^{11}_{(l,q)} = -3Q^4_{(l,q)} + [\frac{80}{9}n_u + \frac{20}{9}n_d + \frac{20}{3}n_l]Q^2_{(l,q)} \\
&\gamma^{13}_{(q)} = -4Q^2_{(q)} \\
&\gamma^{33}_{(q)} = -\frac{404}{3} + \frac{40}{9}(n_u + n_d) \\
&\gamma^{333}_{(q)} = \frac{2}{3}[\frac{140}{27}(n_u + n_d)^2 + (100\zeta(3) + \frac{2216}{9})(n_u + n_d) - 3747] ,
\end{aligned}
\tag{A 4 5}
$$

where $Q_{(l,q)}$ is the electric charge of a given lepton or quark, and $\zeta(3) = 1.2020\ldots$ is the Riemann zeta function evaluated at three. In the mass anomalous dimension for the quarks above, we have also included the three-loop pure QCD contribution $\gamma^{333}_{(q)}$ [93]

APPENDIX B
THE MSSM β FUNCTIONS

Using some of the notation of Falck [90], the superpotential and soft symmetry breaking potential are as follows:

$$W = \bar{u}\mathbf{Y}_u\Phi_u Q + \bar{d}\mathbf{Y}_d\Phi_d Q + \bar{e}\mathbf{Y}_e\Phi_d L + h.c. + \mu\Phi_u\Phi_d ,$$

$$\begin{aligned} V_{soft} = {} & m_{\phi_u}^2\Phi_u^\dagger\Phi_u + m_{\phi_d}^2\Phi_d^\dagger\Phi_d + m_Q^2 Q^\dagger Q + m_L^2 L^\dagger L \\ & + m_u^2 u^\dagger u + m_d^2 d^\dagger d + m_e^2 e^\dagger e + B\mu(\Phi_u\Phi_d + h.c.) \\ & + \sum_{i,j}\left(A_u^{ij}\mathbf{Y}_u{}^{ij}u_i\Phi_u Q_j + A_d^{ij}\mathbf{Y}_d{}^{ij}d_i\Phi_d Q_j + A_e^{ij}\mathbf{Y}_e{}^{ij}\bar{e}_i\Phi_d L_j + h.c.\right) \end{aligned}$$

$$V_{gaugino} = +\frac{1}{2}\sum_{i=1}^{3} M_i\lambda_i\lambda_i + h.c.$$

(B.1)

Various σ_2's have been omitted and a sum over the number of generations is implied. Also hats imply superfields (like the superpartners of the given fields) and overbars the charge conjugate.

B.1 Gauge Couplings

First the gauge couplings are

$$\begin{aligned} \frac{dg_i}{dt} = {} & -\frac{1}{16\pi^2}b_i g_i^3 \\ & - \frac{g_i^3}{(16\pi^2)^2}\left[\sum_k b_{ik}g_k^2 - \mathrm{Tr}\left\{C_{iu}\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + C_{id}\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + C_{ie}\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\right\}\right] \end{aligned}$$

(B.1.1)

where $t = \ln\mu$ and $l = 1\ \ 2\ \ 3$, corresponding to gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ of the standard model. The various coefficients are defined to be

$$b_1 = -\frac{3}{5} - 2n_g \ ,$$
$$b_2 = 5 - 2n_g \ , \qquad \text{(B 1 2)}$$
$$b_3 = 9 - 2n_g$$

$$(\tilde{b}_{lk}) = \begin{pmatrix} \frac{38}{15} & \frac{6}{5} & \frac{88}{15} \\ \frac{2}{5} & 14 & 8 \\ \frac{11}{15} & 3 & \frac{68}{3} \end{pmatrix} n_g + \begin{pmatrix} \frac{9}{25} & \frac{9}{5} & 0 \\ \frac{3}{5} & -17 & 0 \\ 0 & 0 & -54 \end{pmatrix} , \qquad \text{(B 1 3)}$$

and

$$(C_{lf}) = \begin{pmatrix} \frac{26}{5} & \frac{14}{5} & \frac{18}{5} \\ 6 & 6 & 2 \\ 4 & 4 & 0 \end{pmatrix} \quad \text{with } f = u\ ,\ d\ ,\ e\ . \qquad \text{(B 1 4)}$$

with $n_g = \frac{1}{2}n_{fl}$

B 2 Superpotential Parameters μ and the Yukawas

In the following we list the β functions for the parameters of the superpotential

$$\frac{d\ln\mu}{dt} = \frac{1}{16\pi^2}[\ \mathrm{Tr}\{3\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + 3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} - 3(\frac{1}{5}g_1^2 + g_2^2)\ ] \qquad \text{(B 2 1)}$$

In the Yukawa sector the β functions are

$$\frac{d\mathbf{Y}_{u\,d\,e}}{dt} = \mathbf{Y}_{u\,d\,e}(\frac{1}{16\pi^2}\beta_{u\,d\,e}^{(1)} + \frac{1}{(16\pi^2)^2}\beta_{u\,d\,e}^{(2)}) \ , \qquad \text{(B 2 2)}$$

where the one-loop contributions are given by

$$\begin{aligned}
\beta_u^{(1)} &= 3\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + \mathbf{Y}_d{}^\dagger\mathbf{Y}_d + 3\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} - (\frac{13}{15}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2) \\
\beta_d^{(1)} &= 3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_u{}^\dagger\mathbf{Y}_u + \mathrm{Tr}\{3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} - (\frac{7}{15}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2) \ , \\
\beta_e^{(1)} &= 3\mathbf{Y}_e{}^\dagger\mathbf{Y}_e + \mathrm{Tr}\{3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} - (\frac{9}{5}g_1^2 + 3g_2^2) \ ,
\end{aligned} \qquad \text{(B 2 3)}$$

and the two-loop contributions are given by

$$\begin{aligned}
\beta_u^{(2)} = & -4(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 - 2(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 - 2\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u - 9\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\}\mathbf{Y}_u{}^\dagger\mathbf{Y}_u \\
& - \mathrm{Tr}\{3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\}\mathbf{Y}_d{}^\dagger\mathbf{Y}_d - 3\,\mathrm{Tr}\{3(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 + \mathbf{Y}_d{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} \\
& + (\frac{2}{5}g_1^2 + 6g_2^2)\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + (\frac{2}{5}g_1^2)\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + (\frac{4}{5}g_1^2 + 16g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} \\
& + (\frac{26}{15}n_g + \frac{403}{450})g_1^4 + (6n_g - \frac{21}{2})g_2^4 + (\frac{32}{3}n_g - \frac{304}{9})g_3^4 \\
& + g_1^2g_2^2 + \frac{136}{15}g_1^2g_3^2 + 8g_2^3g_3^2
\end{aligned}$$

$$\begin{aligned}
\beta_d^{(2)} = & -4(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 - 2(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 - 2\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\mathbf{Y}_d{}^\dagger\mathbf{Y}_d - 3\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\}\mathbf{Y}_u{}^\dagger\mathbf{Y}_u \\
& - 3\,\mathrm{Tr}\{3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\}\mathbf{Y}_d{}^\dagger\mathbf{Y}_d \\
& - 3\,\mathrm{Tr}\{3(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 + (\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2 + \mathbf{Y}_d{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} + (\frac{4}{5}g_1^2)\mathbf{Y}_u{}^\dagger\mathbf{Y}_u \\
& + (\frac{4}{5}g_1^2 + 6g_2^2)\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + (-\frac{2}{5}g_1^2 + 16g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} \\
& + (\frac{6}{5}g_1^2)\,\mathrm{Tr}\{\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} + (\frac{14}{15}n_g + \frac{7}{18})g_1^4 + (6n_g - \frac{21}{2})g_2^4 \\
& + (\frac{32}{3}n_g - \frac{304}{9})g_3^4 + g_1^2g_2^2 + \frac{8}{9}g_1^2g_3^2 + 8g_2^2g_3^2
\end{aligned}$$

$$\begin{aligned}
\beta_e^{(2)} = & -4(\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2 - 3\,\mathrm{Tr}\{3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d + \mathbf{Y}_e{}^\dagger\mathbf{Y}_e\}\mathbf{Y}_e{}^\dagger\mathbf{Y}_e \\
& - 3\,\mathrm{Tr}\{3(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 + (\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2 + \mathbf{Y}_d{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\} \\
& + (6g_2^2)\mathbf{Y}_e{}^\dagger\mathbf{Y}_e + (\frac{6}{5}g_1^2)\,\mathrm{Tr}\{\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\} + (-\frac{2}{5}g_1^2 + 16g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} \\
& + (\frac{18}{5}n_g + \frac{27}{10})g_1^4 + (6n_g - \frac{21}{2})g_2^4 + \frac{9}{5}g_1^2g_2^2
\end{aligned} \tag{B.2.4}$$

B 3 Higgses' Vacuum Expectation Values

The evolution of the vacuum expectation values of the Higgs's is given by

$$\frac{d\ln v_{\phi_u,\phi_d}}{dt} = \frac{1}{16\pi^2}\gamma^{(1)}_{\phi_u,\phi_d} + \frac{1}{(16\pi^2)^2}\gamma^{(2)}_{\phi_u,\phi_d}, \tag{B 3 1}$$

where the one-loop contribution is given by

$$\begin{aligned}\gamma^{(1)}_{\phi_u} &= \frac{3}{4}(\frac{1}{5}g_1^2 + g_2^2) - 3\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\}\\ \gamma^{(1)}_{\phi_d} &= \frac{3}{4}(\frac{1}{5}g_1^2 + g_2^2) - 3\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} - \mathrm{Tr}\{\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\},\end{aligned} \tag{B 3 2}$$

and the two-loop contribution is given by

$$\begin{aligned}\gamma^{(2)}_{\phi_u} = \frac{3}{4}&\,\mathrm{Tr}\{3(\mathbf{Y}_u{}^\dagger\mathbf{Y}_u)^2 + 3\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\mathbf{Y}_l{}^\dagger\mathbf{Y}_d\}\\ &-(\frac{19}{10}g_1^2 + \frac{9}{2}g_2^2 + 20g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_u{}^\dagger\mathbf{Y}_u\}\\ &-(\frac{279}{800} + \frac{1803}{1600}n_g)g_1^4 - (\frac{207}{32} + \frac{357}{64}n_g)g_2^4 - (\frac{27}{80} + \frac{9}{80}n_g)g_1^2g_2^2,\\ \gamma^{(2)}_{\phi_d} = \frac{3}{4}&\,\mathrm{Tr}\{3(\mathbf{Y}_d{}^\dagger\mathbf{Y}_d)^2 + 3\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\mathbf{Y}_u{}^\dagger\mathbf{Y}_u + (\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^2\}\\ &-(\frac{2}{5}g_1^2 + \frac{9}{2}g_2^2 + 20g_3^2)\,\mathrm{Tr}\{\mathbf{Y}_d{}^\dagger\mathbf{Y}_d\} - (\frac{9}{5}g_1^2 + \frac{3}{2}g_2^2)\,\mathrm{Tr}\{\mathbf{Y}_e{}^\dagger\mathbf{Y}_e\}\\ &-(\frac{279}{800} + \frac{1803}{1600}n_g)g_1^4 - (\frac{207}{32} + \frac{357}{64}n_g)g_2^4 - (\frac{27}{80} + \frac{9}{80}n_g)g_1^2g_2^2\end{aligned} \tag{B 3 3}$$

B 4 Soft Symmetry Breaking Terms

The soft supersymmetry breaking terms are

$$\begin{aligned}\frac{dA_E^{ij}}{dt} = \frac{1}{16\pi^2}\{&4(\mathbf{Y}_e\mathbf{Y}_e{}^\dagger)^{ik}A_e^{kj}\lambda^{kj}_{ijk} + 5A_e^{ik}\lambda^{ik}_{ijk}(\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^{kj} - 3\frac{A_E^{ij}}{\lambda^{ij}_{ijk}}(\mathbf{Y}_e\mathbf{Y}_e{}^\dagger\mathbf{Y}_e)^{ij}\\ &+2(A_e^{km}[\mathbf{Y}_e^{km}]^2 + 3A_d^{km}[\mathbf{Y}_d^{km}]^2 - 6(\frac{3}{5}g_1^2M_1 + g_2^2M_2)\},\end{aligned} \tag{B 4 1}$$

$$\frac{dA_d^{ij}}{dt} = \frac{1}{16\pi^2}[\,4(\mathbf{Y}_d\mathbf{Y}_d^\dagger)^{ik}A_d^{kj}\frac{\lambda_d^{lj}}{Y_d^{lj}} + 5A_d^{ik}\frac{\lambda_d^{lk}}{Y_d^{lj}}(\mathbf{Y}_d^\dagger\mathbf{Y}_d)^{kj} - 3\frac{A_d^{ij}}{Y_d^{ij}}(\mathbf{Y}_d\mathbf{Y}_d^\dagger\mathbf{Y}_d)^{ij}$$
$$+ (A_d^{ik} - A_d^{ij})(\mathbf{Y}_u^\dagger\mathbf{Y}_u)^{kj}\frac{\lambda_d^{jk}}{Y_d^{ij}} + 2(\mathbf{Y}_d\mathbf{Y}_u^\dagger)^{ik}A_d^{kj}\frac{Y_u^{kj}}{Y_d^{ij}}$$
$$+ 2(A_e^{km}|\lambda_e^{km}|^2 + 3A_d^{km}|\lambda_d^{km}|^2) - \frac{14}{15}g_1^2M_1 - 6g_2^2M_2 - \frac{32}{3}g_3^2M_3\,]\,,$$

$$\frac{dA_u^{ij}}{dt} = \frac{1}{16\pi^2}[\,4(\mathbf{Y}_u\mathbf{Y}_u^\dagger)^{ik}A_u^{kj}\frac{\lambda_u^{kj}}{\lambda_u^{ij}} + 5A_u^{ik}\frac{Y_u^{ik}}{\lambda_u^{ij}}(\mathbf{Y}_u^\dagger\mathbf{Y}_u)^{kj} - 3\frac{A_u^{ij}}{Y_u^{ij}}(\mathbf{Y}_u\mathbf{Y}_u^\dagger\mathbf{Y}_u)^{ij}$$
$$+ (A_u^{ik} - A_u^{ij})(\mathbf{Y}_d^\dagger\mathbf{Y}_d)^{kj}\frac{\lambda_u^{ik}}{\lambda_u^{ij}} + 2(\mathbf{Y}_u\mathbf{Y}_d^\dagger)^{ik}A_d^{kj}\frac{Y_d^{kj}}{\lambda_u^{ij}}$$
$$+ 6\,A_u^{km}|\lambda_u^{km}|^2 - \frac{26}{15}g_1^2M_1 - 6g_2^2M_2 - \frac{32}{3}g_3^2M_3\,]$$

$$\frac{dm_{\phi_u}^2}{dt} = \frac{1}{8\pi^2}\{\sum_{i,j} 3|\lambda_u^{ij}|^2(m_{\phi_u}^2 + m_Q^{i\,2} + m_{u_r}^{j\,2} + |A_u^{ij}|^2)$$
$$+ \frac{3}{10}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{3}{5}g_1^2M_1^2 - 3g_2^2M_2^2\,]\,,$$

$$\frac{dm_{\phi_d}^2}{dt} = \frac{1}{8\pi^2}\{\sum_{i,j}(|\lambda_d^{ij}|^2(m_{\phi_d}^2 + m_l^{i\,2} + m_{e_r}^{j\,2} + |A_e^{ij}|^2)$$
$$+ 3|\lambda_d^{ij}|^2(m_{\phi_d}^2 + m_Q^{i\,2} + m_{d_j}^{j\,2} + |A_d^{ij}|^2))$$
$$- \frac{3}{10}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{3}{5}g_1^2M_1^2 - 3g_2^2M_2^2\,]\,,$$

$$\frac{dm_{e_r}^2}{dt} = \frac{1}{8\pi^2}\{\sum_j 2|\lambda_e^{ij}|^2(m_{\phi_d}^2 + m_l^{i\,2} + m_{l_j}^{i\,2} + |A_e^{ij}|^2)$$
$$+ \frac{3}{5}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{12}{5}g_1^2M_1^2\,]$$

$$\frac{dm_{l_i}^2}{dt} = \frac{1}{8\pi^2}\{\sum_j |\lambda_e^{ij}|^2(m_{\phi_d}^2 + m_l^{i\,2} + m_{e_j}^{2} + |A_e^{ij}|^2)$$
$$- \frac{3}{10}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{3}{5}g_1^2M_1^2 - 3g_2^2M_2^2\,]\,,$$

(B.4.2)

$$\frac{dm_{d_i}^2}{dt} = \frac{1}{8\pi^2}\Big[\sum_j 2|Y_d^{ij}|^2(m_{d_j}^2 + m_{H_1}^2 + m_{Q_j}^2 + |A_d^{ij}|^2)$$

$$+ \frac{1}{5}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{4}{15}g_1^2M_1^2 - \frac{16}{3}g_3^2M_3^2\Big]\,,$$

$$\frac{dm_{u_i}^2}{dt} = \frac{1}{8\pi^2}\Big[\sum_j 2|Y_u^{ij}|^2(m_{u_j}^2 + m_{H_2}^2 + m_{Q_j}^2 + |A_u^{ij}|^2)$$

$$- \frac{2}{5}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{16}{15}g_1^2M_1^2 - \frac{16}{3}g_3^2M_3^2\Big]\,,$$

$$\frac{dm_{Q_i}^2}{dt} = \frac{1}{8\pi^2}\Big[\sum_{i,j}(|Y_u^{ji}|^2(m_{u_j}^2 + m_{H_2}^2 + m_{Q_i}^2 + |A_u^{ji}|^2)$$

$$+ |Y_d^{ji}|^2(m_{d_j}^2 + m_{H_1}^2 + m_{Q_i}^2 + |A_d^{ji}|^2))$$

$$+ \frac{1}{10}g_1^2\,\mathrm{Tr}\{Ym^2\} - \frac{1}{15}g_1^2M_1^2 - 3g_2^2M_2^2 - \frac{16}{3}g_3^2M_3^2\Big]$$

$$\frac{dB}{dt} = \frac{1}{8\pi^2}\Big[3A_d^{ij}|Y_d^{ij}|^2 + 3A_u^{ij}|Y_u^{ij}|^2 + A_e^{ij}|Y_e^{ij}|^2 + \frac{4}{5}g_1^2M_1 - 3g_2^2M_2\Big] \tag{B.4.3}$$

where as in Falck [96] sums are implied over all indices not appearing on the left hand side and where

$$\mathrm{Tr}\{Ym^2\} = \sum_{i=1}^{n_g}(m_{Q_i}^2 - 2m_{u_i}^2 + m_{d_i}^2 - m_{L_i}^2 + m_{e_i}^2) \tag{B.4.4}$$

B.5 Gaugino Masses

The gaugino masses evolve as follows

$$\frac{d\ln M_i}{dt} = -\frac{1}{8\pi^2}b_i g_i^2 \tag{B.5.1}$$

APPENDIX C
EXPLICIT FORM OF $\delta(\mu)$

In Ref. 41 the radiative corrections term $\delta(\mu)$, from Eq. (2.2.43), is derived. In this appendix we present its explicit form as it appears in this reference except for some minor notational changes. In the following, s and c refer to $\sin\theta_W$ and $\cos\theta_W$ respectively. Also ξ is defined to be the ratio M_H^2/M_Z^2.

$$\delta(\mu) = \frac{G_\mu M_Z^2}{\sqrt{2}\,8\pi^2}\{\xi f_1(\xi,\mu) + f_0(\xi,\mu) + \xi^{-1} f_{-1}(\xi,\mu)\} \tag{C.1}$$

where the various functions are defined as follows

$$\begin{aligned}
f_1(\xi,\mu) &= 6\ln\frac{\mu^2}{M_H^2} + \frac{3}{2}\ln\xi - \frac{1}{2}Z(\frac{1}{\xi}) - Z(\frac{c^2}{\xi}) - \ln c^2 + \frac{9}{2}(\frac{25}{9} - \frac{\pi}{\sqrt{3}}) \\
f_0(\xi,\mu) &= -6\ln\frac{\mu^2}{M_Z^2}[1 + 2c^2 - 2\frac{M_t^2}{M_Z^2}] + \frac{3c^2\xi}{\xi - c^2}\ln\frac{\xi}{c^2} + 2Z(\frac{1}{\xi}) \\
&\quad + 4c^2 Z(\frac{c^2}{\xi}) + \frac{3c^4\ln c^2}{s^2} + 12c^4\ln c^2 - \frac{15}{2}(1 + 2c^2) \\
&\quad - 3\frac{M_t^2}{M_Z^2}[2Z(\frac{M_t^2}{M_Z^2\xi}) + 4\ln\frac{M_t^2}{M_Z^2} - 5] \\
f_{-1}(\xi,\mu) &= 6\ln\frac{\mu^2}{M_Z^2}[1 + 2c^4 - 4\frac{M_t^2}{M_Z^2}] - 6Z(\frac{1}{\xi}) - 12c^4 Z(\frac{c^2}{\xi}) - 12c^4\ln c^2 \\
&\quad + 8(1 + 2c^4) + 24\frac{M_t^2}{M_Z^2}[\ln\frac{M_t^2}{M_Z^2} - 2 + Z(\frac{M_t^2}{M_Z^2\xi})] ,
\end{aligned} \tag{C.2}$$

with

$$\begin{aligned}
Z(z) &= \begin{cases} 2\,4\tan^{-1}(1/A) & (z > \frac{1}{4}) \\ 4\ln[(1+A)/(1-A)] & (z < \frac{1}{4}) \end{cases} , \\
A &\equiv |1 - 4z|^{\frac{1}{2}}
\end{aligned} \tag{C.3}$$

APPENDIX D
A SELF-DUAL SOLUTION IN 2+1 GRAVITY

We study some consequences of duality in 3 dimensions. In the case of pure gravity a dual Ansatz is shown to lead to pure gauge configurations [25].

D.1 Introduction

In four Euclidean dimensions self duality of Yang Mills gauge fields leads to the classical instanton solutions. A similar condition imposed on the gravitational connection leads to the Eguchi Hanson gravitational instantons. Motivated by the recent interest in theories of lesser number of dimensions [97,98,99] we investigate analogs of self duality in the classical solutions of some theories in 3 (or 2+1) dimensions. In the pure gravitational case we study the analog of the Eguchi Hanson Ansatz [100] which is known to provide a classical solution [101]. It is shown that in a pure gauge (in the sense of Witten [97]) with large gauge configuration which simplifies the study of quantum fluctuations around it.

D.2 Euclidean Gravity in 3 Dimensions

Gravity in the first order formalism is described in terms of the vielbein and connection one forms $e^a(x)$ and $\omega^{ab}(x) = -\omega^{ba}(x)$ respectively under the

SO(n,1) tangent space group then transform as

$$\begin{aligned} \delta e^a &= \epsilon^a{}_b\, e^b \\ \delta\omega^{ab} &= \mathrm{d}\epsilon^{ab} + \epsilon^a{}_c\,\omega^{cb} + \epsilon^b{}_c\,\omega^{ac} \end{aligned} \tag{D 2 1}$$

where $\epsilon^{ab}(=-\epsilon^{ba})$ are the parameters of SO(n,1). The ensuing geometry is built from the following covariant two-forms: the torsion

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b. \tag{D 2.2}$$

and the curvature

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{D 2 3}$$

In an attempt to generalize instanton solutions to Euclidean gravity, Eguchi and Hanson [100] proposed the following Ansatz involving the connection one-form

$$\omega^{-b} = \pm\frac{1}{2}\epsilon^{ab}{}_{cd}\omega^{cd} \tag{D.2 4}$$

which is, of course, patterned to four dimensions, and can only be implemented in Euclidean space. This Ansatz was shown to solve the vacuum Einstein equations, and leads to gravitational instantons, although without an asymptotically flat structure.

Some time ago in an attempt in finding solutions in higher dimensions, generalization of this type of Ansatz was considered in different number of dimensions [101] the rule being that it should involve only the vielbein and the spin connection one-forms (to written in form languages and be manifestly covariant in the tangent space indices, using the flat metric η^{ab} and the relevant Levi Civita tensor. In dimensions higher than four such Ansatz led to $\det e = 0$. In two dimensions the Ansatz

$$\omega = \epsilon^a{}_b\, e^b \tag{D 2 5}$$

consistent only in Minkowski 1+1 space, also leads to $\det e = 0$. Another possible Ansatz

$$\epsilon_{ab}\, \omega^{ab} = 0 \tag{D.2.6}$$

sets the connection to zero. However in 3 dimensions, one can require

$$e^a = \frac{1}{2} l\, \epsilon^a{}_{bc}\, \omega^{bc}, \tag{D.2.7}$$

valid in both Euclidean and Minkowski space, with l a dimensionful constant. In the following, we study the consequences of this Ansatz.

In 3 dimensions gravity can be described in a number of ways. The traditional way is

$$S = \int d^3x\, (\det e)\, [R + \Lambda] \tag{D.2.8}$$

where R is the scalar curvature

$$R = R^{\mu\nu}{}_{\mu\nu}\, E^\mu_a\, E^\nu_b \tag{D.2.9}$$

and E^μ_a is the inverse metric

$$E^i_a\, e^a_\mu = \delta^b_a \tag{D.2.10}$$

In form language, it reads

$$S = \int R^a \wedge e_a + \Lambda \int e^a \wedge e^b \wedge e^c\, \epsilon_{abc} \tag{D.2.11}$$

where

$$R^a = \frac{1}{2}\, \epsilon^a{}_{bc}\, R^{bc} \tag{D.2.12}$$

Furthermore the cosmological term can be rewritten in terms of the torsion, yielding

$$S = \int (R^a + \Lambda\, T^a) \wedge e_a \tag{D.2.13}$$

or as

$$S = \int [(T^a + \Lambda e^a) \wedge e_a - \omega^a \wedge \omega^b \wedge e^c \epsilon_{abc}], \qquad \text{(D 2 14)}$$

where $\omega^a = \frac{1}{2}\, \epsilon^a{}_{bc}\, \omega^{bc}$ showing the ubiquitous role the torsion plays in three dimensions (a fact well known to Fresnel and Serret). At any rate, in the first order formalism, we have

$$\begin{aligned} S = &- \int d^3x\, \epsilon^{\mu\nu\rho} e_{a\rho} (\partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} - \epsilon_{abc} \omega_{\mu b} \omega_{\nu c}) \\ &+ \frac{\Lambda}{3} \int d^3x\, \epsilon^{\mu\nu\rho} \epsilon_{abc} e^a{}_\mu e^b{}_\nu e^c{}_\rho, \end{aligned} \qquad \text{(D 2 15)}$$

where we have chosen to be in Euclidean space and have set $1/16\pi G = 1$ and set ourselves in S^3. We can now look for the equations of motion (We don't have to worry about surface terms when integrating by parts since S^3 has no boundary.)

i) Variation with respect to $e_{\rho a}$: $\frac{\delta S}{\delta e_{\rho a}} = 0$ gives Einstein's equations

$$\partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} - \epsilon_{abc} (\omega_{\mu b} \omega_{\nu c} + \Lambda e_{\mu b} e_{\nu c}) = 0 \qquad \text{(D 2 16)}$$

ii) Variation with respect to $\omega_{\rho a}$: $\frac{\delta S}{\delta \omega_{\rho a}} = 0$ gives the no-torsion condition

$$\partial_\mu e_{\nu a} - \partial_\nu e_{\mu a} - \epsilon_{abc} (\omega_{\mu b} e_{\nu c} - \omega_{\nu b} e_{\mu c}) = 0 \qquad \text{(D 2.17)}$$

Besides the trivial solution $e = \omega = 0$ there are other solutions of interest. Plugging our Ansatz yields the equations

$$\begin{cases} l^{-1} \partial_\mu e_{\nu a} - l^{-1} \partial_\nu e_{\mu a} - \epsilon_{abc} (l^{-2} e_{\mu b} e_{\nu c} + \Lambda e_{\mu b} e_{\nu c}) = 0 \\ \partial_\mu e_{\nu a} - \partial_\nu e_{\mu a} - \epsilon_{abc} l^{-1} (e_{\mu b} e_{\nu c} - e_{\mu b} e_{\nu c}) = 0 \end{cases} \qquad \text{(D 2 18)}$$

Multiplying the second of these equations by l^{-1} and subtracting it from the first equation one obtains

$$\epsilon_{abc} (l^{-2} - \Lambda) e_{\mu b} e_{\nu c} = 0 \qquad \text{(D 2 19)}$$

Therefore $e_a = l\omega_a$ is a solution if $l^2 = \Lambda^{-1}$. Consequently the cosmological constant has to be positive for this solution to exist. Hence our classical solution is

$$e_a = \frac{\omega_a}{\sqrt{\Lambda}} \tag{D.2.20}$$

Note that this Ansatz for a solution also works in the Minkowskian case requiring this time however, that the cosmological constant be negative.

Our next step is to evaluate the action for this particular classical solution. We find

$$\begin{aligned} S^{(0)} = &- \int d^3x\, \epsilon^{\mu\nu\rho} e_{\mu a} (l^{-1} \partial_\rho e_{\nu a} - l^{-1} \partial_\nu e_{\rho a} - \epsilon_{abc} l^{-2} e_{\rho b} e_{\nu c}) \\ &+ \frac{l^{-1}}{3} \int d^3x\, \epsilon^{\mu\nu\rho} \epsilon_{abc} e_{\mu a} e_{\rho b} e_{\nu c} \end{aligned} \tag{D.2.21}$$

Using the no-torsion condition one gets

$$\begin{aligned} S^{(0)} &= (-l^{-2} + \frac{l^{-2}}{3}) \int d^3x\, \epsilon^{\mu\nu\rho} \epsilon_{abc} e_{\mu a} e_{\rho b} e_{\nu c} \\ &= -\frac{4}{3} l^{-1} \int_{M^3} d^3x\, \sqrt{g} \end{aligned} \tag{D.2.22}$$

Therefore

$$S^{(0)} = -\frac{4}{3} \sqrt{\Lambda} \mathrm{vol}(M^3) \tag{D.2.23}$$

The value of the action is finite and therefore suggests an instanton-like nature for our solution. It is again different from the regular four dimensional gravitational instanton solutions where the spin connection is self dual which is possible a priori since it is a two-form and so is its dual. Clearly, in three dimensions we cannot have this sort of solution. However, we may hope to relate the dual, in the flat (tangent) indices of the spin connection to the dreibein. This will be a gauge dependent choice since they do not transform in the same

way We'll see a natural explanation for this behavior in a Chern Simons approach to the theory

In fact let us show that e is of the form $\phi d\phi^{-1}$ where

$$\phi \quad M^3 \cong S^3 \longrightarrow SU(2) \cong S^3 \tag{D.2.24}$$

Such maps are classified homotopically by $\pi_3(SU(2)) = \pi_3(S^3) = Z$, the group of integers Thus implies that $\int_{M^3} d^3x \det e$ will have different values in each homotopically different sector of ϕ

Indeed, the no-torsion condition reads

$$de_a + \omega_{ab} \wedge e_b = de_a + \epsilon_{abc}\omega_c \wedge e_b = 0 \tag{D.2.25}$$

but $\omega_c = \lambda^{-1} e_c$ hence

$$de_a + \lambda^{-1}\epsilon_{abc} e_c \wedge e_b = 0 \tag{D.2.26}$$

If one uses the following anti hermitean realization of the commutation relations of SU(2)

$$\left[\frac{\sigma_a}{2i}, \frac{\sigma_b}{2i}\right] = \epsilon_{abc}\left(\frac{\sigma_c}{2i}\right) \tag{D.2.27}$$

and multiplies Eq. (D.2.26) by $\sigma_a/2i$ one obtains

$$d\left(\frac{\sigma_a e_a}{2i}\right) - 2i^{-1}\left(\frac{\sigma_b e_b}{2i}\right) \wedge \left(\frac{\sigma_c e_c}{2i}\right) = 0 \tag{D.2.28}$$

Therefore $e = \sigma_a e_a/2i$ must satisfy

$$de - 2i^{-1} e \wedge e = 0 \tag{D.2.29}$$

Let's try a solution of the form $e = \xi d\phi\phi^{-1}$ then

$$de = \xi d\phi \wedge d\phi^{-1} \tag{D.2.30}$$

and

$$
\begin{aligned}
-\omega_{ab} &= -\xi^2 \sigma \mathrm{d}\sigma^{-1} \wedge \phi \mathrm{d}\phi^{-1} \\
&= \xi^2 \phi(-\phi^{-1}\mathrm{d}\phi) \wedge \mathrm{d}\phi^{-1} \\
&= -\xi^2 \mathrm{d}\phi \wedge \mathrm{d}\phi^{-1},
\end{aligned}
\tag{D 2.31}
$$

so that

$$
\mathrm{d}e - 2l^{-1} e \wedge e = (\xi + 2l^{-1}\xi^2)\mathrm{d}\phi \wedge \mathrm{d}\phi^{-1} = 0 \tag{D 2.32}
$$

This will be satisfied non trivially if $\xi = -l/2$ therefore the dreibein can be written as

$$
e = -\frac{l}{2}\phi \mathrm{d}\phi^{-1} = \frac{-1}{2\sqrt{\Lambda}}\phi \mathrm{d}\phi^{-1} \tag{D 2.33}
$$

We can describe the map ϕ as follows

$$
\begin{aligned}
\phi \quad M^1 \cong S^3 &\longrightarrow SL(2) \cong S^3 \\
x &\longmapsto \sigma(x) = n_0(x) + i\vec{\sigma}\cdot\vec{n}(x)
\end{aligned}
\tag{D 2.34}
$$

Then $\phi^{-1}(x) = n_0(x) - i\vec{\sigma}\cdot\vec{n}(x)$ if $n_0^2(x) + |\vec{n}(x)|^2 = 1$, so that indeed $\phi(x) \in S^3$

The integer q classifying homotopically the maps $S^3 \xrightarrow{\phi} S^3$ is called the degree of ϕ and is given by the following expression

$$
\begin{aligned}
q &= \frac{-1}{24\pi^2}\int_{M^3} d^3x\, \varepsilon^{\mu\rho\gamma}\, \mathrm{Tr}(\sigma\partial_i\sigma^{-1}\sigma\partial_\nu\sigma^{-1}\sigma\partial_\rho\phi^{-1}) \\
&= \frac{-1}{2\pi^2 l^3}\int_{M^3} d^3x \det e = \frac{-\Lambda}{2\pi^2 l}\mathrm{Vol}(M^4)
\end{aligned}
\tag{D 2.35}
$$

Therefore Eq. (D 2.23) becomes

$$
S^{(0)} = +8\pi^2 l q = +8\pi^2 \frac{q}{\sqrt{\Lambda}} \tag{D.2.36}
$$

The study of this solution as plays-a-role in scaling and its quantum fluctuation is immensely simplified by Witten's reformulation [97] of 3 dimensional gravity

D 3 Chern Simons Description of 2+1 Gravity

Consider the Chern Simons action associated with a gauge group G, it is given by

$$I_{CS}(A) = \frac{1}{2}\int_{M^3} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \qquad \text{(D 3 1)}$$

$$= \frac{1}{4}\int_{M^3} d^3x\ \epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) + \frac{2}{3}\left[A_\mu\ A_\nu\right] A_\rho\right) \qquad \text{(D 3 2)}$$

Here $A = A_\mu dx^\mu = A_a T^a$, the T^a s are the generators of the Lie algebra associated to the gauge group G that one uses depending on the theory to be described.

To recover 2+1 gravity with positive cosmological constant Λ, Witten [97] has shown that one should use the group of symmetries of de Sitter space in 3 dimensions, namely SO(3,1) as the appropriate gauge group in the Lorentzian metric case. Then in terms of the generator of the Lie algebra $so(3,1)$ P^a for the translations and $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$ for the Lorentz generators one can write $A = e^a P_a + \omega^a J_a = e^a P_a - \frac{1}{2}\omega^{ab}J_{ab}$

Witten [97] also argued that to describe the Euclidean version of the theory one should change the gauge group from SO(3,1) to SO(4) since in gravity it represents the tangent space group. This step is somewhat ambiguous since the argument in the exponential in the path integral will still have an overall factor of "i" which because SO(4) is compact will imply, as we shall see later, that the action must be quantized. In fact this only happens in the case of *Euclidean* gravity with *positive* cosmological constant, since in all other cases the gauge group would not be compact. If one wants to get rid of this "i", so as to conform with the regular four dimensional approach, one would need

to rotate the dreibein e into ω. We won't consider this "Euclideanization" in here. With this a proviso our $so(4)$ Lie algebra valued gauge field becomes

$$A = e_a P_a - \frac{1}{2}\omega_{ab} J_{ab} = e_a P_a - \omega_a J_a$$

Here P_a and J_a are the antihermitean generators of the $so(4) \cong su(2) \oplus su(2)$ Lie algebra and obey the following commutation relations

$$\begin{aligned} [J_a\ J_b] &= \epsilon_{abc} J_c \\ [J_a\ P_b] &= \epsilon_{abc} P_c \\ [P_a\ P_b] &= \Lambda \epsilon_{abc} J_c \end{aligned} \tag{D 3 3}$$

The trace in Eq (D 3 1) is well defined because there exists (in three dimensions only) the invariant bilinear form $W = P_a J_a$ (up to normalization) which allow for the existence of the following non degenerate quadratic form (or metric) on our Lie algebra.

$$\langle J_a\ P_b \rangle = 2\delta_{ab}$$

$$\langle J_a\ J_b \rangle = 0 = \langle P_a\ P_b \rangle \tag{D 3 4}$$

As noted in [97] there exists another natural quadratic form when the cosmological constant is non-zero. This is due in our case to the semisimplicity of $so(4)$ which gives $\pi_3(SO(4)) = Z + Z$ and therefore allows a priori for two possible Chern-Simons terms with different couplings. However, only the choice described in the text recovers our original action; in other words we are taking this allowed second coupling parameter to be zero. Therefore the trace that we shall be using is defined by

$$\begin{aligned} \mathrm{Tr}\,(P_a J_b) &= \mathrm{Tr}\,(J_a P_b) = 2\delta_{ab} \\ \mathrm{Tr}\,(J_a J_b) &= 0 = \mathrm{Tr}\,(P_a P_b) \end{aligned} \tag{D.3.5}$$

Let's use these relations to rewrite Eq. (D.3.2)

$$I_{CS} = \frac{1}{4}\int_{M^3} d^3x\, \epsilon^{\mu\nu\rho}\Big(-2\epsilon_{\mu a}(\partial_\nu\omega_{\rho a} - \partial_\rho\omega_{\nu a}) - 2\omega_{\mu a}(\partial_\nu e_{\rho a} - \partial_\rho e_{\nu a})\Big)$$
$$+\frac{1}{4}\int_{M^3} d^3x\, \epsilon^{\mu\nu\rho}\frac{2}{3}\mathrm{Tr}\Big(\{\epsilon_{\mu a}\epsilon_{ab}\Lambda\epsilon_{bdc}J_c - \epsilon_{\mu a}\omega_{b[}\epsilon_{ab]c}P_c - \omega_{\mu a}e_{b[}\epsilon_{ab]c}P_c$$
$$+\omega_{\mu a}\omega_{a[}\epsilon_{b]dc}J_c\}(e_{\rho[}P_{d]} - \omega_{\rho[}J_{d]})\Big) \tag{D.3.6}$$

Integrating by part the second term in the first integral and taking the traces in the second integral one recovers our action Eq. (D.2.15)

$$I_{CS} = S = -\int d^3x\, \epsilon^{\mu\nu\rho}\epsilon_{mn}(\partial_\mu\omega_{\nu n} - \partial_\nu\omega_{\mu n} - \epsilon_{nbc}\omega_{\mu b}\omega_{\nu c})$$
$$+\frac{1}{3}\int d^3x\, \epsilon^{\mu\nu\rho}\epsilon_{abc}\epsilon_{mn}e_{\mu b}e_{\nu c} \tag{D.3.7}$$

D.4 Perturbative Evaluation of the Partition Function

To evaluate the partition function around our classical solution $e = \omega/\sqrt{\Lambda}$ in the semiclassical approximation (and in fact also exactly) we can use the results of Witten [98] and Chamseddine [99], modified to apply to our different gauge group. But first let us change our notation slightly exploiting the fact that $so(4) \cong su(2) + su(2)$. If one introduces the following linear combination of our generators

$$J_a^{\pm} = \frac{1}{2}\left(J_a \pm \frac{1}{\sqrt{\Lambda}}P_a\right)$$

then the commutation relations can be rewritten as

$$[J_a^{\pm}\ J_b^{\pm}] = \epsilon_{abc}J_c^{\pm}$$
$$[J_a^{+}\ J_b^{-}] = 0 \tag{D.4.1}$$

which makes the splitting explicit. Now correspondingly we can define $A_a^{\pm} = -\omega_a \pm \sqrt{\Lambda}e_a$, then

$$A = A_a^{+}J_a^{+} + A_a J_a^{-} = A^{+} + A^{-} \tag{D.4.2}$$

Since the two $su(2)$ commute, the Chern Simons action Eq (D 3 1) is just

$$I_{CS}(A) = I_{CS}(A^+) + I_{CS}(A^-) \tag{D 4 3}$$

where

$$I_{CS}(A^\pm) = \frac{1}{2}\int_{M^3} \mathrm{Tr}\left(A^\pm \wedge dA^\pm + \frac{2}{3}A^\pm \wedge A^\pm \wedge A^\pm\right) \tag{D 4 4}$$

Now "Tr" means

$$\mathrm{Tr}(J^\pm_a J^\pm_b) = \frac{\pm 1}{\sqrt{\Lambda}}\delta_{ab} \tag{D 4 5}$$

Let a change the normalization of the trace to conform with the standard normalization used in Chern Simons theories namely one takes the trace in the fundamental representation of the group Therefore introduce Tr' such that

$$\mathrm{Tr}'(J^\pm_a J^\pm_b) = -\frac{1}{2}\delta_{ab}$$

Then (D 4 4) becomes

$$I_{CS}(A^\pm) = \frac{\mp 1}{\sqrt{\Lambda}}\int_{M^3} \mathrm{Tr}'\left(A^\pm \wedge dA^\pm + \frac{2}{3}A^\pm \wedge A^\pm \wedge A^\pm\right) = \mp\frac{I'_{CS}(A^\pm)}{\sqrt{\Lambda}} \tag{D 4 6}$$

The partition function will therefore be

$$\begin{aligned} Z &= \int \mathcal{D}A\, e^{i\sqrt{\Lambda}(I_{CS}(A^+) - I_{CS}(A^-))} \\ &= \int \mathcal{D}A^+ \mathcal{D}A^- \exp\left(ik^+ I_{CS}(A^+) + ik^- I'_{CS}(A^-)\right), \end{aligned} \tag{D 4 7}$$

where we have introduced $k^\pm = \mp 4\pi/\sqrt{\Lambda}$

Since A^+ and A^- are two commuting independent gauge fields, we must have

$$Z = Z^+ Z^- \tag{D 4 8}$$

where

$$Z^{\pm} = \int \mathcal{D}\mathbb{A}^{\pm} \exp\left(\frac{ik^{\pm}}{4\pi}\Gamma_{CS}\left(\mathbb{A}^{\pm}\right)\right) \qquad (D\,4\,9)$$

In the semiclassical approximation for large $k^{\pm}$ or small $\mathbb{A}$, we can evaluate $Z^{\pm}$ around any classical solution $\mathbb{A}_0^{\pm}$. These extrema of the action correspond to the flat connections, i.e., solutions of

$$\mathbb{F} = d\mathbb{A} + \mathbb{A}\wedge\mathbb{A} = 0 \qquad (D\,4\,10)$$

which is just a way of rewriting our equations of motion Eq. (D 2 16,D 2.17).

Up to a possible overall phase factor, the quadratic fluctuations, gauge fixing and ghosts contributions give us, as in Witten [98] or Carlip and de Alwis [99]

$$Z^{\pm} \sim (k^{\pm})^{\frac{1}{2}(b^1 - b^0)} \int_{H^1/\mathcal{G}_0} \mathrm{T}\left(\mathbb{A}_0^{\pm}\right) \exp\left(\frac{ik^{\pm}}{4\pi}\Gamma_{CS}\left(\mathbb{A}_0^{\pm}\right)\right) \qquad (D\,4\,11)$$

where H^0, H^1 are the set of zero modes of the covariant Laplaceans made of $\mathbb{A}_0^{\pm}$ acting on functions and differential forms of first order respectively, with corresponding dimension b^0 and b^1, and $\mathrm{T}(\mathbb{A}_0^{\pm})$ is the square root of the Ray Singer analytic torsion.

The integral over H^1 is an integral over gauge equivalence classes of flat connections. The integral over $(H^0)^{-1}$ is the ghost zero-modes integral which means that one must divide out the volume of the isotropy group, i.e. the group that leaves points on the manifold unmoved when acting on it.

Now we wish to evaluate $Z^{\pm}$ around the particular flat connection that we have described earlier, namely $e = \frac{\mathbb{A}}{\sqrt{\Lambda}}$ or, in a different notation,

$$\mathbb{A}_0^{+} = 0 \quad \text{and} \quad \mathbb{A}_0^{-} = -2\sqrt{\Lambda}\, e_0 \qquad (D\,4\,12)$$

Let us show that in fact this solution can be obtained by a "large" (not connected to the identity) gauge transformation from the trivial flat connection $A = 0$, i.e., $e = \omega = 0$

A "large" gauge transformation of $A = 0$ is of the form UdU^{-1}, with $U(x) \in SO(4)$ We have

$$A_0 = A_{0a}^{+} J_a^{+} + A_{0a}^{-} J_a^{-} = -2\sqrt{\lambda}\, e_{0a} J_a^{-} = -2\sqrt{\lambda}\, e_0 \qquad (D\,4\,13)$$

But from Eq (D 2 33) $e_0 = \frac{1}{2\sqrt{\lambda}} \phi d\phi^{-1}$ Therefore $A_0 = \phi d\phi^{-1}$ which indeed implies that A_0 is a "large" gauge transform of $A = 0$ In fact A_0 and $A = 0$ are in the same equivalence class of pure gauge fields in our case because our manifold is S^3 Indeed (as in Witten [98]) gauge equivalence classes of such flat connections correspond to equivalence classes up to conjugation of homomorphisms

$$\tau_1(M^3) \xrightarrow{f} G \qquad (D\,4\,14)$$

But since $M^3 = S^3$ and $\tau_1(S^3) = 0$ there is only one trivial class of such homomorphisms and so for any gauge group G

Therefore we can take over the results for the expansion of the partition function around the trivial flat connection and apply them to our case, the only difference residing in the fact that our gauge groups are different since we are using Euclidean signature metrics

Our purpose now is to evaluate Eq (D 4 11) for our particular solution Since A_0 is gauge equivalent to $A=0$ the Ray Singer analytical torsion is just the inverse of the measure of the isotropy group which for S^3 is the group of *global* gauge transformations namely $SU(2)$ For $A_0^{+}=0$ we obviously have $I_{CS}(A_0^{+} = 0) = 0$, but since the Chern Simons form is not gauge invariant we don t have $I_{CS}(A_0^{-}) = 0$ It is usually proportional to the degree of $M^3 \xrightarrow{\phi} G$

however, we have already done the calculation when we computed $S^{(0)}$ as given by Eq. (D 2 36). Indeed

$$S^{(3)} = I_{CS}(A_\theta) = \frac{k^-}{4\pi} Y_{CS}(A_0^-) = 8\pi^2 \frac{\theta}{\sqrt{\Lambda}} \tag{D 4 15}$$

However, we need to require that e^{iS} be gauge invariant which therefore enforces a quantization condition on the cosmological constant Λ. Precisely we must have $\frac{8\pi^2}{\sqrt{\Lambda}} = 2\pi \times$ integer or $k^- = \frac{1\pi}{\sqrt{\Lambda}}$ must be an integer, so that $e^{iS^{(3)}} = 1$.

Since there is only one class of flat connections, the trivial one, the integral in Eq. (D 4 11) over H^1 is absent and $b^1 = 0$. From what we have said about the Ray-Singer torsion, the remaining integral just picks up the inverse of the volume of SU(2) which is just a constant and we have $b^0 = 3$.

Therefore the large $k^\pm$ limit (in semiclassical approximation) for $Z^\pm$ is, up to a phase

$$Z^\pm \sim |k^\pm|^{-3/2} \tag{D 4.16}$$

so that for the partition function Z one has

$$Z \sim |k^+ k^-|^{-3/2} = \Lambda^{3/2} \tag{D 4 17}$$

D.5 Exact Evaluation of the Partition Function

In fact one can do better and compute the partition function exactly. Indeed from Eq. (D 4 8)

$$\begin{aligned} Z(S^3,\ SO(4)) &= Z(S^3,\ SU(2) \times SU(2)) \\ &= Z^+(S^3,\ SU(2))\, Z^-(S^3,\ SU(2)) \end{aligned} \tag{D 5 1}$$

But $Z(S^3, SU(2))$ has been computed by Witten[98] up to a phase it is given by

$$Z^\pm(S^3,\ SU(2)) = \sqrt{\frac{2}{k^\pm + 2}} \sin\left(\frac{\pi}{k^\pm + 2}\right) \tag{D 5.2}$$

Therefore the full partition function is

$$Z(S^3,\ SO(4)) = \frac{2}{\sqrt{k^+ + 2}\sqrt{k^- + 2}} \sin\left(\frac{\pi}{k^+ + 2}\right) \sin\left(\frac{\pi}{k^- + 2}\right) \tag{D 5 3}$$

In the large $k^{\pm}$ limit we recover

$$Z(S^3,\ SO(4)) \sim k^{3/2} \tag{D 5 4}$$

Having the closed formula for the partition function we can using the methods of Ref 98, compute the expectation value of generally covariant and gauge invariant observables namely Wilson lines. They are defined as follows: given an oriented, non self intersecting circle $\mathcal{C}$ in S^3 (an unknot), and an irreducible representation R of G we can compute the holonomy of A around $\mathcal{C}$ and take its trace in the representation R: the resulting quantity is a Wilson line

$$W_R(\mathcal{C}) = \mathrm{Tr}_R P \exp \oint_{\mathcal{C}} A \tag{D 5 5}$$

In our case, A takes value in $so(2) \sim u(2)$ where the two $su(2)$ commute, then

$$\begin{aligned} W_{R_+R_-}(\mathcal{C}) &= \mathrm{Tr}_{R_+ R_-} P \exp \oint_{\mathcal{C}} (A^+ + A^-) \\ &= \mathrm{Tr}_{R_+} \exp \oint_{\mathcal{C}} A^+ \ \mathrm{Tr}_{R_-} \exp \oint_{\mathcal{C}} A^- \\ &= W_{R_+}(\mathcal{C}) W_{R_-}(\mathcal{C}) \end{aligned} \tag{D 5 6}$$

The expectation value of such a Wilson line is then

$$\begin{aligned} Z(S^3, \mathcal{C}, R_{+-}) &= \int \mathcal{D}A\ W_R(\mathcal{C}) e^{ik\mathrm{CS}(A)} \\ &= \int \mathcal{D}A^+ \mathcal{D}A^- W_{R_+}(\mathcal{C}) W_{R_-}(\mathcal{C}) e^{ik\mathrm{CS}(A^+)} e^{-ik\mathrm{CS}(A^-)} \\ &= Z^+(S^3, \mathcal{C}, R_+)\, Z^-(S^3, \mathcal{C}, R_-) \end{aligned} \tag{D 5 7}$$

Using Eq (D 5 1), we can rewrite this as

$$\frac{Z(S^3, \mathcal{C}, R_{+-})}{Z(S^3)} = \frac{Z^+(S^3, \mathcal{C}, R_+)}{Z^+(S^3)} \ \frac{Z^-(S^3, \mathcal{C}, R_-)}{Z^-(S^3)} \tag{D 5 8}$$

The various quantities in this equation have been computed in Ref. 98. If R is the spin j representation of SU(2) then

$$< C_j >= \frac{Z(S^3\ C_R_j)}{Z(S^3)} = \frac{S_{0,j}}{S_{0,0}}, \tag{D.5.9}$$

where S generates modular transformation on the torus and is known explicitly from the theory of affine Lie algebras

$$S_{0,j} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(j+1)\pi}{k+2}\right) \tag{D.5.10}$$

Therefore we can rewrite (D.5.8) as

$$\begin{aligned} < C_{j^+} - > =& < C_{j^+} >< C_{j^-} > \\ =& \frac{S_{0,j^+}}{S_{0,0}} \frac{S_{0,j^-}}{S_{0,0}} \\ =& \frac{\sin(\frac{(j^++1)\pi}{k^++2})}{\sin(\frac{\pi}{k^++2})} \frac{\sin(\frac{(j^-+1)\pi}{k^-+2})}{\sin(\frac{\pi}{k^-+2})} \end{aligned} \tag{D.5.11}$$

We can then compute the expectation value of two unknotted, unlinked Wilson lines and obtain

$$\begin{aligned} < C_{j_1^+} - C_{j_2^+} > =& \frac{Z(S^3\ C_{j_1^+} \quad C_{j_2^+} -)}{Z(S^3)} \\ =& < C_{j_1^+}\ C_{j_1^+} >< C_{j_1^-}\ C_{j_2^-} > \end{aligned} \tag{D.5.12}$$

But from [98] we know that

$$< C_{j_1^\pm}\ C_{j_2^\pm} > = < C_{j_1^\pm} >< C_{j_2^\pm} > \tag{D.5.13}$$

therefore (D.5.12) becomes

$$< C_{j_1^+} - C_{j_2^+} > = < C_{j_1^+} >< C_{j_2^+} >< C_{j_1^-} >< C_{j_2^-} > \tag{D.5.14}$$

In this manner we can evaluate the expectation value of an arbitrary number of unlinked, unknotted Wilson lines on S^3 carrying the spin j representation of SU(2) in terms of the expectation value of just one Wilson line

$$< C_{j_1^+} \quad C_{j_r^+} - > = \prod_{i=1}^{r} < C_{j_i^+} >< C_{j_r^-} > \tag{D.5.15}$$

Then for an arbitrary combination of Wilson lines one would need to use the skein relation which describe how to pass a Wilson line through another in order to bring the problem back to unknotted circles

Let us add some comments about the special nature of Chern Simons theories with gauge group SO(4) The Lie group SO(4) is the only compact, semisimple Lie group which is not simple This means at the level of the Lie algebra for example that $so(4)$ is the sum of two simple Lie algebras, namely two $su(2)$ as mentioned earlier This unique fact entails that $\pi_2(SO(4)) = Z+Z$ and therefore allows two possible non degenerate invariant quadratic forms on the $so(4)$ Lie algebra Consequently one should in the most general case, consider exhibiting the two possible terms that will arise when one writes down the associated Chern Simons action However each term will come with its own coupling constant k_1 and k_2 and, following arguments identical to the ones described earlier one would conclude that the two coupling constants get quantized Taking these facts into account would give a supplementary contribution to the partition function and would accordingly modify the observables of the theory namely the expectation values of Wilson lines as well as the skein relations needed to compute them

Another interesting case is when the Lie algebra does not possess a non degenerate invariant quadratic form Weyl theory is such an example in 2+1 dimensions The group of symmetry that one would consider in a Chern-Simons approach should be the Weyl group namely the Poincaré group plus the dilatations However one cannot build an invariant quadratic form on the associated Lie algebra Therefore it would seem that one cannot describe Weyl theory in 2+1 dimensions as a Chern Simons theory However, let us note in passing that if one includes the special conformal transformations then one

is dealing with the full conformal group in 2+1 dimensions, i.e., SO(3,2) and Horne and Witten [102] have shown that this group can be used successfully in a description a *la* Chern Simons of conformal gravity. However, this was achieved by going in a special gauge. It may be therefore possible to find a gauge choice that would describe Weyl theory in 2+1 dimensions.

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BIOGRAPHICAL SKETCH

The author was born in the south of France, at La Seyne-Sur-Mer, near Toulon on January 24 1961. He was educated mostly in France in Paris or its suburbs. At the University of Paris Jussieu in 1983 he received various *maitrises* in physics, mathematics and applied mathematics. He decided to continue his education in the United States, first at Columbia University in New York City for two years, where he earned a Master of Philosophy in Physics in 1985. Before transferring to the University of Florida in 1986 he fulfilled his military obligations for his native country during the academic year 1985-1986. He came here to pursue his interest in particle theory. He has endeavored to maintain a strong interest in the phenomenological as well as in the more theoretical aspects of his field. He is currently working on extending this work to supergravity inspired minimal extension of the standard model. He hopes to be able to continue to teach and do his research for years to come starting with a postdoctoral position in the physics department of the University of Virginia in Charlottesville.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy

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